Part III: Surfaces

curves

\[ r(t) \]

- parametrization: \( r(t) \)
- tangent vector \( r'(t) \)
- curvature

surfaces

\[ r(u,v) \]

tangent plane (complicated)
1) curvature
2) length of curve
3) line integral of vector field (work)
4) orientation (direction)
5) fundamental theorem of line integrals

(complicated)
area of surface

function

surface integrals of field (flux)

orientation (up/down)

Stoke's theorem
divergence theorem

How to parameterize a surface:
surface is two dimensional → need two variables
surface is two dimensional \[ \Rightarrow \] need two variables
surface is in 3d space \[ \Rightarrow \] output is a 3d vector

$$\mathbf{r}(u_0, v_0)$$

Example:

$$\mathbf{r}(\varphi, \theta) = \langle \sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi \rangle$$

$$0 \leq \varphi \leq 2\pi$$
We compute that
\[ x^2 + y^2 + z^2 = \]

\[ 0 \leq \varphi \leq 2\pi \]
\[ 0 \leq \vartheta \leq \frac{\pi}{2} \]

\[ \varphi \]... latitude (measured from north pole)
\[ \vartheta \]... longitude (measured
\[ x + y + z = \]
\[ = (\sin \varphi \cos \theta)^2 + (\sin \varphi \sin \theta)^2 + \cos \varphi \]
\[ = \sin^2 \varphi (\cos^2 \theta + \sin^2 \theta) + \cos \varphi \]
\[ = \sin^2 \varphi + \cos^2 \varphi = 1 \]

\[ \Rightarrow \text{upper hemisphere of the unit sphere.} \]

Another parametrization of the upper hemisphere:

\[
\begin{align*}
x &= u \\
y &= v \\
z &= \sqrt{1 - u^2 - v^2}
\end{align*}
\]

\[ D = \{(u,v) : u^2 + v^2 \leq 1\} \]
\[ = \text{unit disk in the } uv \text{-plane.} \]
\[ z = \sqrt{1 - u^2 - v^2} \]

= unit disk in the $uv$-plane.

Generally, the graph for the function $z = f(x,y)$...
generally, a graph for a plane can always be parametrized as
\[ \mathbf{r}(u, v) = \langle u, v, f(u, v) \rangle \]

Ex: Find a parametrization for the plane through \((x_0, y_0, z_0)\) containing the vectors \(\mathbf{A}\) and \(\mathbf{B}\).

Line: point \((x, y, z)\), vector: \(\mathbf{A}\)
\[ \mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\mathbf{A} \]
\[ \vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \hat{A} \]

\[ \vec{r}(u, v) = \langle x_0, y_0, z_0 \rangle + u \hat{A} + v \hat{B} \]

\( (u, v) \in \text{plane} \)

**Ex:** Find a parametrization of the triangle with vertices \((0,0,1), (1,0,0), (0,1,0)\)

\[ \vec{r}(u,v) = \langle 1, 0, 0 \rangle + u \langle -1, 1, 0 \rangle + v \langle -1, 0, 1 \rangle = \langle 1-u-v, u, v \rangle \]
\[
\mathbf{r}(u,v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi
\]

\[
D = \{(u,v) : u,v > 0, \ u+v \leq 1\}
\]

\[
\mathbf{r}(1-u-v, u, v) = \langle 1-u-v, u, v \rangle
\]
helical ramp

$u, v$ are polar coordinates

(u ... radius)

(v ... angle)

from above: