Last time: if \( \vec{F} = \nabla f \), then \( \int_C \vec{F} \cdot d\vec{r} \) is independent of the path.

\[ \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \]

If we integrate around a closed loop, then we get:

\[ \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{C_2} = \int_{C_1} - \int_{C_2} = \]
\[ \int \vec{F} \cdot d\vec{r} = \int_{C_1} + \int_{-C_2} = \int_{C_1 - C_2} = \int_C \]

\( \oint \) means that \( C \) is a loop.

Converse statement:

If \( \vec{F} \) is defined and continuous on all of \( \mathbb{R}^2 \) and \( \int_C \vec{F} \cdot d\vec{r} \) only depends on the end points of \( C \), then there exists a function \( f \) with \( \nabla f = \vec{F} \) (i.e. \( \vec{F} \) is conservative).
Proof: We have to somehow cook up a candidate for the potential function $f$. 

\[ f(x, y) = \int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{s} \]

By path independence, this does not depend on the chosen path.

Write $\vec{F} = \langle P, Q \rangle$. We have to show that 

$\nabla f = \vec{F}$, i.e. that $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$.

To do that, let us choose a nice path for the integral defining $f(x,y)$:
the integral defining \( f(x, y) \):

\[
f(x, y) = \int_{C_1} P \, dx + Q \, dy + \int_{C_2} P \, dx + Q \, dy
\]

\( C_1 \)
\( C_2 \)

\[
does \, not \, depend \, on \, x, \, so \, \frac{2}{\partial x} \int_{C_1} = 0
\]

\[
y \, is \, constant
\]

\[
\dot{r}(t) = \langle t, y \rangle
\]

\[
\dot{r}'(t) = \langle 1, 0 \rangle
\]

\[
\int_{0}^{x} P(t, y) \, dt
\]

so

\[
\frac{2}{\partial x} \int_{C_1 + C_2} P \, dx + Q \, dy = \frac{2}{\partial x} \int_{0}^{x} P(t, y) \, dt
\]
\( C_1 + C_2 + \oint_0 \) \\
\( = P(x, y) \)

so 
\[ \frac{\partial}{\partial x} f(x, y) = P(x, y) \]

Similarly, to compute \( \frac{\partial}{\partial y} f \) we choose the path 

and with a similar argument to before we compute 
\[ \frac{\partial}{\partial y} f = Q(x, y) \]

So 
\[ \nabla f = \vec{F} \], as was required. \( \square \)
(Side note: Here we are cheating a bit and assume that \( \vec{F} \) is defined everywhere. Otherwise one just chooses paths that avoid the places where \( f \) is not defined.)

**Summary:**

\[ \vec{F} \text{ is conservative } (\vec{F} = \nabla f) \quad \iff \quad \vec{F} \text{ has the path independence property} \]

The path independence property is impossible to check in practice.

To find a simple way to test whether
We need a simple way to test whether a given \( \vec{F} \) is conservative.

If \( \vec{F} = \nabla f \),

\[
P \text{\( \frac{\partial f}{\partial x} \text{\( \hat{i} \)} + \frac{\partial f}{\partial y} \text{\( \hat{j} \)} \), i.e.} \quad P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}
\]

\[
P_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = Q_x
\]

So

\[
\vec{F} = P \text{\( \hat{i} \)} + Q \text{\( \hat{j} \)} \quad \Rightarrow \quad P_y = Q_x
\]

is conservative
In other words if \( P_y \neq Q_x \), then \( \vec{F} \) is not conservative.

The reverse implication is only true if the domain of \( \vec{F} \) is simply connected (= has no holes).

Thus:

\[ P_y = Q_x \quad \text{and} \quad \text{domain of } \vec{F} \text{ is simply connected} \]

\[ \Rightarrow \quad \vec{F} = P_x + Q_y \quad \text{is conservative} \]

Ex: \( \) Is \( \vec{F} = \langle xe^y, ye^x \rangle \) conservative?
\[ P_y = \frac{\partial P}{\partial y} = xe^y, \quad Q_x = ye^x \]

not the same, so not conservative.

\(1\) \(F = \langle ye^x + \sin y, e^x + x \cos y + e^y \rangle\)

\[ P = Q \]
\[ P_y = e^x + \cos y \]
\[ Q_x = e^x + \cos y \]

\(F\) is defined everywhere (domain of \(F\) is \(\mathbb{R}^2\) is simply connected) and \(P_y = Q_x\) so \(\bar{F}\) is conservative.

Let's compute a potential \(f\), i.e. \(f_x = P, f_y = Q\)

\[ \frac{\partial f}{\partial y} = ye^x + \sin y, \quad f(x,y) = ye^x + x \sin y + q(y) \]
\[ \frac{\partial f}{\partial x} = ye^x + \sin y \quad \Rightarrow \quad f(x, y) = ye^x + x\sin y + g(y) \]
\[ \frac{\partial f}{\partial y} = e^x + x\cos y + g'(y) = e^x + x\cos y + e^y \]
\[ g'(y) = e^y \]
\[ g(y) = e^y + C \]

So \[ f(x, y) = ye^x + x\sin y + e^y + C \] has \[ \nabla f = \vec{F}. \]

\[ \begin{array}{c}
\text{Ex: Is } \vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \text{ conservative?} \\
\quad P \quad Q \\
P = -1(x^2+y^2) + y(2y) = -x^2 + y^2 \\
\end{array} \]
\[ p_y = \frac{-x^2 \cdot y^3 + y(2y)}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} \]

\[ q_x = \frac{x^2 - y^2 - x \cdot (2x)}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} \]

\( \vec{F} \) is not defined at \((0,0)\), so we cannot use the criterion to check whether it is conservative. Let's compute an integral around a closed loop around the origin.

The domain of \( \vec{F} \) has a hole.

\[ \vec{r}(t) = \langle R \cos t, R \sin t \rangle \]
\[ \vec{F}(t) = \langle R \cos t, R \sin t \rangle \]

\[ \vec{v}'(t) = \langle -R \sin t, R \cos t \rangle \]

\[ \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-R \sin t}{R^2}, \frac{R \cos t}{R^2} \right\rangle \cdot \vec{v}'(t) dt \]

\[ = \cdots = 2\pi \quad \text{interestingly this does not depend on } R \]

So \( \vec{F} \) is not conservative.