Practice midterm & learning expectations are online.

Anyone here not registered?

Last time: line integrals of vector fields:

\[
\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} \, ds = \int_{t=a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]
\[ F = \langle p, q \rangle : \quad = \int_C P \, dx + Q \, dy \]

Of course the same definitions also work in three dimensions.

In general, for two curves \( C_1 \) and \( C_2 \),

\[ \int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}, \]

\( C_1, \quad C_2 \)
even if they have the same start and end points.

The vector field near $C_1$ has nothing to do with the vector field near $C_2$.

Amazing Theorem:

Suppose $\vec{F} = \nabla f$ (i.e. $\vec{F}$ is conservative) and $C$ is a curve from $p_1$ to $p_2$. Then:
C is a curve from \( p_1 \) to \( p_2 \). Then:

\[
\int_C \vec{F} \cdot d\vec{r} = f(p_2) - f(p_1)
\]

In particular, \( \int_C \vec{F} \cdot d\vec{r} \) is independent of the choice of curve.

This is rather surprising: \( f \) (and hence \( \vec{F} \)) near \( C_1 \) has still nothing to do with \( \vec{F} \) near \( C_2 \).
On the other hand the structure of the above theorem should look familiar:

\[ F = \frac{d}{dx} f \]

\[ \int_{a}^{b} F \, dx = f(b) - f(a) \]

\[ \int_{C} \vec{F} \cdot d\vec{r} = f(B) - f(A) \]

Here every cts \( F(x) \) has an antiderivative.

Not every cts vector field \( \vec{F}(x,y,z) \) has a potential.

Proof of the theorem:
\[ \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \]
\[ \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \]
\[ \vec{F} = \nabla f \text{ so } \vec{F} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle \]

\[ \int_C \vec{F} \cdot d\vec{s} = \int_{t_0}^{t_1} \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle \, dt \]

\[ = \int_{t_0}^{t_1} \left( \frac{dx}{dt} \frac{dx}{dt} + \frac{dy}{dt} \frac{dy}{dt} + \frac{dz}{dt} \frac{dz}{dt} \right) \, dt \]

\[ = \left. \frac{d}{dt} \left( f(x(t), y(t), z(t)) \right) \right|_{t_0}^{t_1} \]
\[
\frac{d}{dt} \mathbf{f}(x(t), y(t), z(t)) \, dt = \mathbf{f}(x(t), y(t), z(t)) - \mathbf{f}(x(t_0), y(t_0), z(t_0))
\]

Example: \( \mathbf{F} = \langle 2x, 2y \rangle \)

We have \( \mathbf{F} = \nabla f \) where
\[
f(x, y) = x^2 + y^2 + C
\]

\( C_1: \mathbf{r}(t) = \langle t, 0 \rangle, \quad 0 \leq t \leq 1 \)

Then
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 0) - f(0, 0) = 1
\]

"Old fashioned way": \( \mathbf{r}'(t) = \langle 1, 0 \rangle \), \( t \in [0, 1] \)
\[ \int_{C_1} \vec{F} \cdot d\vec{v} = \int_0^1 \langle 2x, 2y \rangle \cdot \langle 1, 0 \rangle \, dt = \int_0^1 2t \, dt = t^2 \bigg|_0^1 = 1 \]

\[ C_1 \quad t = 0 \quad \uparrow_{x = t} \quad y = 0 \]

\[ C = C_2 + C_3 \quad \vec{r}_2(t) = \langle t, -t \rangle \quad 0 \leq t \leq 1 \quad \vec{r}_3(t) = \langle 0, -1 + t \rangle \]

\[ \int_{C_2 + C_3} \vec{F} \cdot d\vec{v} = \int_{C_2} + \int_{C_3} = \int_0^1 \langle 2x, 2y \rangle \cdot \langle 1, -1 \rangle \, dt + \int_0^1 \langle 2x, 2y \rangle \cdot \langle 0, 1 \rangle \, dt \]

\[ C_2 \quad c_2 \quad c_3 \quad 0 \quad \uparrow_{x = t} \quad \downarrow_{y = -1} \quad 0 \quad \uparrow_{x = 0} \quad \uparrow_{y = t - 1} \]

\[ = \int_0^1 2t + 2t \, dt + \int_0^1 2t - 2 \, dt = 2t^2 \bigg|_0^1 + (t^2 - 2t) \bigg|_0^1 = 2 + 1 - 2 = 1 \]