Equipotential = level curve of potential

eg. \( \vec{F}(x,y) = \langle x, y \rangle \), \( f(x,y) = \frac{1}{2} (x^2 + y^2) \) ... potential \( (\nabla f = \vec{F}) \)

\[ \Rightarrow f(x,y) = C \] ... level curves/equipotentials

Line integrals of vector fields

\[ \vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j} \]

\( C \) a curve.
How would we integrate such a thing?
What do we even want to get?

Let's get some inspiration from physics.

Work done to push a box a distance \( D \) using a force of magnitude \( F \):

\[
W = F \cdot D
\]

Generalize this to vectors and we have a way to interpret the integral of a vector field (= force) along a curve (= distance).
In 2/3 dimensions only the component of \( \vec{F} \) in the direction of displacement contributes to the work done.

\[
\vec{F} = \text{force of wind}
\]

\[
\vec{D} = \text{displacement}
\]

Work: \( W = \vec{F} \cdot \vec{D} \)

Here the displacement was along a straight line and the force was constant. More generally we would move along a curve through a vector field.
As usual: split up the curve and approximate using segments & constant force:

\[ \mathbf{r}(t_i + \Delta t) - \mathbf{r}(t_i) \approx \mathbf{r}'(t) \Delta t \]

\[ = \frac{\mathbf{r}'(t_i)}{|\mathbf{r}'(t_i)|} \cdot |\mathbf{r}'(t)| \Delta t \]

\[ = \mathbf{T}(t) \Delta s \]

So work done during the \( i \)th segment

\[ \approx \mathbf{F}(x(t_i), y(t_i)) \cdot \mathbf{T}(t) \Delta s \]
\[ W = \lim_{N \to \infty} \sum_{i=0}^{N-1} \hat{F}(x(i), y(i)) \cdot \vec{T}(i) \Delta s \]

\[ = \int_C \vec{F} \cdot d\vec{s} = \int_C \hat{F}(x(t), y(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)| \, dt \]

\[ = \int_{t=a}^{b} \hat{F}(x(t), y(t)) \cdot \vec{r}'(t) \, dt \quad \text{while} \quad \vec{r}'(t) = \frac{d\vec{r}}{dt} \]

\[ = \int_C \vec{F} \cdot d\vec{r} \]

\[ \int_C \vec{F} \cdot d\vec{s} := \int_C \vec{F} \cdot d\vec{s} = \int_C \hat{F}(x(t), y(t)) \cdot \vec{r}'(t) \, dt \]

This depends on the orientation!
So \[ \int_C \vec{F} \cdot \vec{T} \, ds = -\int_{-C} \vec{F} \cdot \vec{T} \, ds. \]

Let's be more explicit: say \[ \vec{F} = P(x,y) \hat{i} + Q(x,y) \hat{j} \]
\[ \vec{r} = \langle x(t), y(t) \rangle \]
\[ \vec{r}' = \langle x'(t), y'(t) \rangle \]

\[ \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{r}' \, dt = \]
\[ = \int_a^b \left( P \frac{dx}{dt} + Q \frac{dy}{dt} \right) \, dt \]
\[ \int_C P \, dx + Q \, dy \]

so

\[ \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} \, ds = \int_C P \, dx + Q \, dy \]

if \( \vec{F} = \langle P, Q \rangle \)

Example: \( C \) - circle of radius 3 oriented counter-clockwise.

\[ \vec{F} = \frac{-y \vec{i}}{\sqrt{x^2 + y^2}} + \frac{x \vec{j}}{\sqrt{x^2 + y^2}} = \frac{\langle -y, x \rangle}{|\langle -y, x \rangle|} \] - unit length

\[ 0 \leq t \leq 2\pi \]

\[ \vec{r}(t) = \langle 3 \cos(t), 3 \sin(t) \rangle \]

\[ \vec{r}'(t) = \langle -3 \sin(t), 3 \cos(t) \rangle \]
\[ \int \vec{F} \cdot d\vec{r} = \int \frac{-y}{\sqrt{x^2+y^2}} \, dx + \frac{x}{\sqrt{x^2+y^2}} \, dy \]

\[ = \int_0^{2\pi} \left( \frac{-3 \sin(t)}{3 \cos^2(t) + 3 \sin^2(t)} \cdot (-3 \sin(t)) + \frac{3 \cos(t)}{3 \cos^2(t) + 3 \sin^2(t)} \cdot (3 \cos(t)) \right) \, dt \]

\[ = \int_0^{2\pi} \frac{9 \sin^2(t)}{3} + \frac{9 \cos^2(t)}{3} \, dt = 3 \int_0^{2\pi} \, dt = 6\pi \]

Same integral: note \( \vec{F} = \vec{T} \), so \( \vec{F} \cdot \vec{T} = 1 \)

\[ \int \vec{F} \cdot \vec{T} \, ds = \int 1 \, ds = 6\pi. \]
In that same example, consider the two semi-circles from \((3,0)\) to \((-3,0)\):

For \(C_1\): \(\dot{F} = F\)

For \(C_2\): \(\dot{F} = -F\)

So \(\int_{C_1} \dot{F} \cdot d\vec{r} = 3\pi\)

\(\int_{C_2} \dot{F} \cdot d\vec{r} = -3\pi\)
Thus in general for two curves $C_1$ and $C_2$,

$$\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r},$$

even if they have the same start and end points.