2nd midterm: Rescheduled to March 8.

Homework: Please add name, student id, course nr (317) and section (201)

Staple!

Recent homework was easier than what you can expect for the midterm. There will be practice midterms on the website.

Line Integrals
Today we'll start to discuss how to integrate functions and vector fields along a curve. Such integrals are called **line integrals**.

Let's start with the 2-d case:

\[ C - \text{parametrized curve given by } \mathbf{r}(t), \ a \leq t \leq b. \]

\[ f(x,y) - \text{function in the plane} \]
We want to integrate $f$ along $C$.

Usual approach: chop the curve up, take a sum, take a limit.

\[ \mathbf{r}(t) = \langle x(t), y(t) \rangle \]

\[ \text{Integral} = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x(t_i), y(t_i)) \Delta s_i \]
\[
\text{Integrate: } \lim_{N \to \infty} \sum_{i=0}^{N-1} f(y(t_i), y'(t_i)) ds.
\]

Recall: \( \Delta s_i = \left| \mathbf{r}(t_{i+1}) - \mathbf{r}(t_i) \right| = \Delta t \left| \frac{\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)}{\Delta t} \right| \\
\approx \Delta t \left| \mathbf{r}'(t_i) \right|
\]

\[
\int_{c} f \, ds = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x(t_i), y(t_i)) \left| \mathbf{r}'(t_i) \right| \Delta t
\]

\[
= \int_{a}^{b} f(\mathbf{r}(t)) \cdot \left| \mathbf{r}'(t) \right| \, dt
\]

\[
= \int_{a}^{b} f(x(t), y(t)) \cdot \sqrt{x'(t)^2 + y'(t)^2} \, dt.
\]

What did we actually compute here?
What did we actually compute here.

Example: Integrate \( f(x,y) = xy^4 \) along the right half of the circle \( x^2 + y^2 = 16 \).
1. Parameterize the curve
\[ \vec{r}(t) = \langle 4 \cos(t) , 4 \sin(t) \rangle \]
\[-\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \]
2. \[ \vec{r}'(t) = \langle -4 \sin(t) , 4 \cos(t) \rangle \]
\[ |\vec{r}'(t)| = 4 \]
3. \[ \int_C xy^4 \, ds = \int_C \frac{d}{dt} \left( 4 \cos(t), 4 \sin(t) \right) \cdot 4 \, dt \]
\[-\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \]
\[ = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cos(t) \cdot 4 \sin^4(t) \, dt = 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4(t) \, dt \]
\[ \sin^4(t) \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2 \cdot \frac{\xi^5}{4} \]
\[
\text{another parametrization: } \vec{r}(t) = \left( \sqrt{16-t^2}, t \right), -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}
\]

\[
|\vec{r}(t)| = \sqrt{\frac{t^2}{16-t^2} + 1} = \sqrt{\frac{16}{16-t^2}} = \frac{4}{\sqrt{16-t^2}}
\]

\[
\int_C xy^4 \, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{16-t^2} \cdot t^4 \cdot \frac{4}{\sqrt{16-t^2}} \, dt
\]
\[ \int_{-5}^{5} 4t^3 \, dt = \frac{4}{5} t^5 \bigg|_{-5}^{5} = \frac{4}{5} (5^5 - (-5)^5) = \frac{2 \cdot 5^2}{5} = \text{same result.} \]

What is \( \int_{-5}^{5} xy \, ds \) for \( C \) this part of the circle?