Recall:

Curvature = \text{change in unit tangent vector as we move along the curve with constant speed 1 (i.e. with respect to the arclength parameter)}

\[ \frac{d \hat{T}}{ds} \]

Chain rule:

\[ \frac{\hat{T}'(t)}{|\hat{r}'(t)|} = \frac{|\hat{r}' \times \hat{r}''|}{|\hat{r}'|^3} \]
Ex: What is the curvature of the graph $y = f(x)$ in the plane?

Find a parametrization:

$$\vec{r}(t) = \langle t, f(t), 0 \rangle$$

$$\vec{r}'(t) = \langle 1, f'(t), 0 \rangle$$
$$\vec{r}''(t) = \langle 0, f''(t), 0 \rangle$$

$$|\vec{r}'(t)| = \sqrt{1 + (f'(t))^2}$$

$$|\vec{r}'(t) \times \vec{r}''(t)| = |\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix}| = |f''(t)\hat{k}| = |f''(t)|$$
What is $\vec{T}'$?

We used the magnitude of $\vec{T}'(t)$ to compute $K(t)$, but what is $\vec{T}'(t)$ geometrically?

Because $|\vec{T}'| = 1$, we have $\vec{T} \perp \vec{T}'$.

$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} = \text{principal/normal vector}$.

$\vec{B} = \vec{N} \times \vec{T}$

$\vec{T}$ tells us the direction we are going.
\( \vec{N} \) tells us the direction we are turning.

**Osculating Circle:**

The circle "kissing" the curve at \( \vec{r}(t) \):

1. tangent to the curve at \( \vec{r}(t) \)
2. radius \( \frac{1}{K(t)} \)
3. lying in the plane spanned by \( \vec{N} \) and \( \vec{T} \).
The osculating circle has the same curvature and normal vector as $\mathbf{C}$ & tangent vector

5. Motion, velocity, acceleration

\[ \mathbf{r}(t) = \text{position vector} \]
\[ \mathbf{r}'(t) = \mathbf{v}(t) = \text{velocity vector} \]
\[ |\mathbf{r}'(t)| = |\mathbf{v}(t)| = \text{speed} = \frac{ds}{dt} \quad (\text{rate of change of arc length}) \]
\[ \mathbf{v}'(t) = \mathbf{r}''(t) = a(t) = \text{acceleration vector} \]

Typical problem: Reconstruct the motion ($\mathbf{r}(t)$) from the acceleration $\mathbf{a}(t)$. 
from the acceleration $\ddot{a}(t)$.

Why do we care? We want to know how some object moves when some force acts on it.

Newton says: force = mass $\cdot$ acceleration

\[ \vec{F}(t) = m \cdot \ddot{a}(t) \]

Example:

\[ \vec{F} = -m \cdot g \cdot \hat{\jmath} \]

so $\ddot{a}(t) = \langle 0, -g, 0 \rangle$

\[ \vec{v}(0) = \vec{v}_0 \]

\[ \vec{r}(0) = \vec{0} \]

\[ \ddot{a}(t) = \vec{v}'(t) = -g \hat{\jmath} \]

\[ \Rightarrow \vec{v}(t) = -gt \hat{\jmath} + \vec{c} \]

\[ \vec{v}(0) = \vec{v}_0 \]
\[ \vec{v}(t) = -g t \hat{j} + \vec{c} \quad \vec{v}(0) = \vec{v}_0 \]
\[ \vec{v}(t) = -g t \hat{j} + \vec{v}_0 \]
\[ \vec{v}(t) = \vec{v}'(t) \]
\[ \vec{r}(t) = -g \frac{t^2}{2} \hat{j} + t \vec{v}_0 + \vec{D} \quad \vec{r}(0) = \vec{0} \]
\[ \vec{D} = \vec{0} \]

so we get \[ \vec{r}(t) = -g \frac{t^2}{2} \hat{j} + t \vec{v}_0 \]
\[ |\vec{v}_0| = v_0 \quad \vec{v}_0 = \langle v_0 \cos \alpha, v_0 \sin \alpha \rangle \]
\[ \vec{r}(t) = \langle tv_0 \cos \alpha, tv_0 \sin \alpha - \frac{g t^2}{2} \rangle, \quad t > 0 \]

When does it hit the ground?
\[ tv_0 \sin \alpha - \frac{g t^2}{2} = 0 \]
\[ t \left( v_0 \sin \alpha - \frac{g t}{2} \right) = 0 \]
\[ t \left( \frac{V_0 \sin \alpha - \frac{g}{2} t}{2} \right) = 0 \]

\[ t = 0 \text{ or } t = \frac{2V_0 \sin \alpha}{g} = t_{end} \]

How far did it go?

\[ \vec{r} (t_{end}) = \left( \frac{2V_0^2 \sin \alpha \cdot \cos \alpha}{g} , 0 \right) \]

Distance = \[ \frac{1}{g} V_0^2 \sin (2\alpha) \]