3. Arc length of a curve

Approximate the curve by $N$ segments.

$t_{start} \leq t \leq t_{end}$

$$\vec{r}(t_{i+1}) - \vec{r}(t_i) = \frac{\vec{r}(t_i + \Delta t) - \vec{r}(t_i)}{\Delta t} \cdot \Delta t$$
\[ \Delta t \approx \vec{r}'(t_i) \cdot \Delta t \]

for small \( \Delta t \)

\[
\text{Length} \approx \sum_{i=0}^{N} |\vec{r}(t_{i+1}) - \vec{r}(t_i)| \approx \sum_{i=0}^{N} |\vec{r}'(t_i)| \Delta t
\]

take limit as \( N \to \infty \) (i.e. \( \Delta t \to 0 \))

\[
\text{Length} = \int_{t_{\text{start}}}^{t_{\text{end}}} |\vec{r}'(t)| \, dt
\]

Assume that the parametrization traverses the curve once.

\( \vec{r}(t) \) - position vector.
\( \vec{r}(t) \) — position vector
\( \vec{r}'(t) \) — velocity vector
\( |\vec{r}'(t)| \) — speed

Distance travelled = integral of speed over time.

Ex: Find the length of one revolution of the helix \( \vec{r}(t) = \langle a \cos t, a \sin t, \frac{bt}{2\pi} \rangle \) (0 ≤ t ≤ 2π)
a, b constants

\( \vec{r}'(t) = \langle -a \sin t, a \cos t, \frac{b}{2\pi} \rangle \)

\[
|\vec{r}'(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + \frac{b^2}{(2\pi)^2}}
\]

\[
= \sqrt{a^2 + \frac{b^2}{(2\pi)^2}}
\]
\[ \text{Length} = \int_{0}^{2\pi} \sqrt{a^2 - \frac{h^2}{(2\pi)^2}} \, dt = 2\pi \sqrt{a^2 + \frac{h^2}{(2\pi)^2}} = \sqrt{(2\pi a)^2 + h^2} \]

**Recall:** There are many different parametrizations for the same curve.

**Question:** Is there a best or canonical parametrization?
Idea: Use the arclength as parameter.

\[ s(t) = \int_{t_0}^{t} |\vec{r}'(u)| \, du \]

Given \( \vec{r}(t) \), \( t_0 \leq t \leq t_1 \), define

\[ s(t) = \text{length of curve from } \vec{r}(t_0) \text{ to } \vec{r}(t). \]

\[ \frac{ds}{dt} = |\vec{r}'(t)| = \text{speed} \]

[FTC] \[ \frac{d}{dx} \int_c^x f(u) \, du = f(x) \]
\[
\frac{ds}{dt} = \left| \vec{r}'(t) \right| = \text{speed}
\]

To reparametrize a curve by arc length:

Take \( \vec{r}(t) \) \((t_0 \leq t \leq t_1)\),

1) Find \( s(t) \)\(^1\):
\[
s(t) = \int_{t_0}^{t} \left| \vec{r}'(u) \right| du = \text{some formula in terms of } t.
\]

2) \( s(t) = \frac{t}{t'\text{'}s} \), solve for \( t(s) \)\(^2\):

3) \( \vec{r}(t) = \vec{r}(t(s)) = \text{function in terms of the arc length } s \).
Example: \( \vec{r}(t) = \langle 3 \sin t, 4t, 3 \cos t \rangle \)

reparametrize in terms of arclength measured from 0.

\[
\vec{r}'(t) = \langle 3 \cos t, 4, -3 \sin t \rangle
\]

\[
|\vec{r}'(t)| = \sqrt{9 \cos^2 t + 16 + 9 \sin^2 t} = \sqrt{9 + 16} = 5
\]

\[
s(t) = \int_0^t 5 \, du = 5t
\]

\[
t(s) = \frac{s}{5}
\]

\[
\vec{r}(s) = \vec{r}(t(s)) = \langle 3 \sin \left(\frac{s}{5}\right), 4 \cdot \frac{s}{5}, 3 \cos \left(\frac{s}{5}\right) \rangle
\]

Example: \( \vec{r}(t) = \langle t^2, \frac{4+12}{3} t^{3/2}, 2t \rangle \) from \( t=0 \)

\[
\vec{r}'(t) = \langle 2t, 2+12 \cdot t^{1/2}, 2 \rangle
\]
\[
|\vec{r}'(t)| = \sqrt{4t^2 + 8t + 4} = 2\sqrt{(t+1)^2} = 2(t+1)
\]

\[
s(t) = \int_0^t |\vec{r}'(u)| \, du = \int_0^t 2u + 2 \, du = \left[ u^2 + 2u \right]_0^t = t^2 + 2t
\]

\[
s = t^2 + 2t
\]

\[
s + 1 = t^2 + 2t + 1
\]

\[
s + 1 = (t + 1)^2
\]

\[
\sqrt{s+1} = t + 1
\]

\[
t = \sqrt{s+1} - 1
\]

\[
\vec{r}(s) = \left( (\sqrt{s+1} - 1)^2, \frac{4\sqrt{2}}{3} (\sqrt{s+1} - 1)^{3/2}, 2(\sqrt{s+1} - 1) \right)
\]