Last time: Examples of parametrized curves.
Today: Let's do some calculus.

2. Derivative of a vector valued function

Def: \( \frac{d\vec{r}}{dt} = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \)
so: if \( \vec{r}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k} \),

then: \( \vec{r}'(t) = f'(t) \hat{i} + g'(t) \hat{j} + h'(t) \hat{k} \).

\( r'(t) \) is tangent to the curve.

Notice: here it is convenient to place the vector \( \vec{r}'(t) \) at the end point of \( \vec{r}(t) \) (rather than at the origin).
Ex: Find the parameterization of the line tangent to the helix \( \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \) at the point \((1, 0, 0)\).

we have \( t = 0 \)
\( \mathbf{r}''(t) = \langle -\sin t, \cos t, 1 \rangle \)
\( \mathbf{r}'(0) = \langle 0, 1, 1 \rangle \)

Thus we want to parameterize the line through \((1, 0, 0)\) in the direction \(\langle 0, 1, 1 \rangle\).

\[ \mathbf{l}(t) = \langle 1, 0, 0 \rangle + t \langle 0, 1, 1 \rangle = \langle 1, t, t \rangle \]

**Derivative rules:**

*additivity:* \( \frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t) \)
additivity: \[
\frac{d}{dt} \left[ \vec{u}(t) + \vec{v}(t) \right] = \vec{u}'(t) + \vec{v}'(t)
\]

product rules:
\[
\frac{d}{dt} \left[ f(t) \vec{v}(t) \right] = f'(t) \vec{v}(t) + f(t) \cdot \vec{v}'(t)
\]

\[
\frac{d}{dt} \left[ \vec{u}(t) \cdot \vec{v}(t) \right] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)
\]

\[
\frac{d}{dt} \left[ \vec{u}(t) \times \vec{v}(t) \right] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)
\]

Proofs: write out \( \vec{u}(t), \vec{v}(t) \) in components and use the usual product rule.
The usual process rule.

Ex: Suppose \( \mathbf{r}(t) \) is a vector-valued function with \( |\mathbf{r}(t)| = 1 \). What can we say about \( \mathbf{r}'(t) \)?

\[
A: \quad |\mathbf{r}(t)| = 1 \iff \mathbf{r}(t) \cdot \mathbf{r}(t) = 1
\]

\[
\implies \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0
\]

\[
\implies 2 \mathbf{r}'(t) \cdot \mathbf{r}(t) = 0
\]

\[
\implies \mathbf{r}'(t) \perp \mathbf{r}(t)
\]

generically:

curve is on the surface of a
surface of a sphere with radius 1.
any vector tangent to a sphere at a point \((x_0, y_0, z_0)\)
is \(\perp\) to the vector \(<x_0, y_0, z_0>\).

Chain rule:

\[
\frac{d}{dt} \left( \vec{r}(f(t)) \right) = f'(t) \vec{r}'(f(t))
\]

Ex:

Find \( \frac{d}{dt} \left( \vec{r}(t) \cdot (\vec{r}'(t) \times \vec{r}''(t)) \right) = \)

\[
\vec{r}''(t) \cdot (\vec{r}'(t) \times \vec{r}''(t)) + \vec{r}'(t) \cdot \left( \vec{r}'(t) \times \vec{r}''(t) \right)'
\]

is \(\perp\) \(\vec{r}'(t)\)
\[ \dot{v}(t) \perp \ddot{v}(t) \]

\[
0 = \dot{v}(t) \cdot \left( \begin{array}{l}
\dddot{v}(t) \times \ddot{v}(t) + \dot{v}(t) \times \dddot{v}(t) \\
= 0
\end{array} \right)
\]

\[
\dot{v}(t) \cdot (\dot{v}(t) \times \dddot{v}(t)).
\]

3. **Arc length of a curve**

\[ t_{\text{start}} \leq t \leq t_{\text{end}} \]

Approximate the curve by \( \vec{r}(t_{\text{start}}) \) to \( \vec{r}(t_{\text{end}}) \).
Approximate the curve by $N$ segments:

$$t_{\text{start}} \leq t_i \leq t_{i+1} \leq t_{\text{end}}$$

$$\vec{r}(t_i) \approx \vec{r}(t_i + \Delta t)$$

$$\vec{r}(t_{i+1}) - \vec{r}(t_i) = \frac{\vec{r}(t_i + \Delta t) - \vec{r}(t_i)}{\Delta t} \cdot \Delta t$$

$$\approx \vec{r}'(t_i) \cdot \Delta t$$

For small $\Delta t$

Length $\approx \sum_{i=0}^{N} |\vec{r}(t_{i+1}) - \vec{r}(t_i)| \approx \sum_{i=0}^{N} |\vec{r}'(t_i)| \Delta t$
Length = \lim_{N \to \infty} \sum_{i=0}^{N} |\hat{v}'(t_i)| \Delta t
= \int_{t_{\text{start}}}^{t_{\text{end}}} |\hat{v}'(t)| \, dt

"Assume that the parametrization traverses the curve once."