Part I. Vector-valued functions of one variable

1. Parametrized Curves

Last time:
- vectors
- vector-valued functions

Ex: \( \mathbf{r}(t) = \langle 1, 0, 0 \rangle + t \cdot \langle 0, 1, 2 \rangle, \quad 0 \leq t \leq 1 \)

\[ x = \frac{1}{2}, \quad \langle 1, \frac{1}{2}, 1 \rangle \]

More generally: we can parametrize a line passing through \((x_0, y_0, z_0)\) moving in the direction of \(\langle a, b, c \rangle\) by
\[ \vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle \]

**Question**: What does \( \vec{v}(t) = \langle 1, t^2, 2t^3 \rangle \), \( 0 \leq t \leq 1 \) parametrize? 

**Answer**: This is the same line segment as above.

Parametrizations are not unique!

In the two examples the particle traverses the same line in the same amount of time, but at different speeds (the second one starts slow and ends fast).
Ex: Unit circle in the plane

Use angle as parameter $t$

$x = \cos t$

$y = \sin t$

$(z = 0)$

$\mathbf{v}(t) = <\cos t, \sin t, 0>$

$\mathbf{v}(t) = <\cos (t^3), \sin (t^3), 0>$, $0 \leq t < 2\pi$

$\mathbf{v}(t) = <\sin t, -\cos t, 0>$, $0 \leq t < 2\pi$

Counterclockwise, starting at bottom:

$\mathbf{v}(t) = <\sin t, \cos t, 0>$, $0 \leq t < 2\pi$

Clockwise, starting at top:

$\mathbf{v}(t) = <\cos t, \sin t, t>$, $0 \leq t \leq 4\pi$

This is a helix.
Ex: Find a parametrization of the curve

\[ \{ z = -\sqrt{x^2 + y^2} \} \cap \{ z = 1 + y \} \]

\( \text{cone} \) \( \text{plane} \)

\[ \begin{align*}
(0, 0, 1) \\
(0, -1, 0) \\
-y + z & = 1
\end{align*} \]

\( \text{normalized vector} \)

\( \langle 0, -1, 1 \rangle \)

Candidates for \( t \) is \( x = t \)

What are \( y \) and \( z \) in terms of \( t \)? Use the equations to solve

\[ z = \sqrt{t^2 + y^2} = 1 + y \]

\[ t^2 + y^2 = 1 + 2y + y^2 \]

\[ t^2 = 1 + 2y, \quad y = \frac{t^2 - 1}{2} \]

\[ z = \frac{t^2 + 1}{2} \]
\[ \vec{r}(t) = \left< t, \frac{t^2 - 1}{2}, \frac{t^2 + 1}{2} \right> \quad -\infty < t < \infty \]

\[ z = \frac{u}{2} \]