Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. Marking scheme: 1 for each correct, 0 otherwise

Consider the function, \( h(x) = 2x^3 - 9x^2 + 12x \).

(a) What are the coordinates of the local maximum of \( h(x) \)?

**Answer:** (1, 5)

**Solution:** The function \( h(x) \) has critical points, but no singular points since its derivative \( h'(x) = 6x^2 - 18x + 12 \) is defined for all values of \( x \). The critical points of \( h(x) \) are computed by equating \( h(x) = 0 \) which yields

\[
6x^2 - 18x + 12 = 0, \text{ i.e. } 6(x^2 - 3x + 2) = 6(x - 1)(x - 2) = 0,
\]

and thus \( x = 1 \) and \( x = 2 \). Using either the Second Derivative Test, i.e. computing \( h''(x) = 12x - 18 \) and then plugging in the critical numbers \( x = 1 \) and \( x = 2 \) in \( h''(x) \), or by simply noticing that \( h'(x) \) changes from positive to negative at \( x = 1 \) and from negative to positive at \( x = 2 \), we conclude that \( x = 1 \) is a point of local maximum, while \( x = 2 \) is a point of local minimum. We compute \( f(1) = 5 \) and \( f(2) = 4 \).

(b) What are the coordinates of the local minimum of \( h(x) \)?

**Answer:** (2, 4)

Short answer questions — you must show your work

2. 8 marks Each part is worth 2 marks.

(a) Find the intervals where \( f(x) = \frac{x^2}{x - 3} \) is decreasing.

**Solution:** First of all, \( f(x) \) is defined for all \( x \neq 3 \). In order to find where is \( f(x) \) decreasing, we find where is \( f'(x) \) negative. So,

\[
f'(x) = \frac{2x(x - 3) - x^2 \cdot 1}{(x - 3)^2} = \frac{2x^2 - 6x - x^2}{(x - 3)^2} = \frac{x(x - 6)}{(x - 3)^2},
\]

and thus, since the denominator is always positive, we conclude that \( f'(x) < 0 \) when \( x(x - 6) < 0 \). Recalling that the domain of definition for \( f(x) \) is \( x \neq 3 \), we conclude that \( f(x) \) is decreasing on the intervals \((0, 3)\) and \((3, 6)\).

**Marking scheme:**

- 1 mark for computing \( f'(x) \) in simplified form, i.e. \( f'(x) = \frac{x(x - 6)}{(x - 3)^2} \).
- 1 mark for writing the correct intervals where \( f(x) \) is increasing; it is acceptable also \([0, 3) \cup (3, 6)\), but **not** acceptable \((0, 6)\).

(b) Let \( f(x) = \cos(x^3 + x^2 - 2) \sin(2x) \). Show that there exists a real number \( c \) such that \( f'(c) = 0 \).
**Solution:** We note that \( f(0) = f(\pi) = 0 \). Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists \( c \in (0, \pi) \) such that
\[
f'(c) = \frac{f(2\pi) - f(0)}{\pi - 0} = 0.
\]
(Alternatively, Rolle’s Theorem gives the same conclusion.)

**Marking scheme:**
- 1 mark for writing that \( f(0) = f(\pi) = 0 \).
- 1 mark for invoking Mean Value Theorem or Rolle’s Theorem correctly to conclude the existence of \( c \in (0, 2\pi) \) such that \( f'(c) = 0 \).

(c) Evaluate \( \lim_{x \to \infty} \frac{\arctan x - \frac{\pi}{2}}{1/x} \).

**Solution:** This limit is a \( \frac{0}{0} \)-indeterminate form with differentiable numerator and denominator. So we can use de L’Hospital’s rule:
\[
\lim_{x \to \infty} \frac{\arctan x - \frac{\pi}{2}}{1/x} = \lim_{x \to \infty} \frac{\frac{1}{1+x^2}}{-x^{-2}} = \lim_{x \to \infty} \frac{-x^2}{1 + x^2} = \lim_{x \to \infty} \frac{-1}{x^2 + 1} = -1.
\]

(d) Evaluate \( \lim_{x \to 0^+} \log(x) \tan(x) \).

**Solution:** This limit is a \( \cdot \) -indeterminate form with differentiable numerator and denominator. So we rewrite it as a \( \cdot \) -indeterminate form:
\[
\lim_{x \to 0^+} \log(x) \tan(x) = \lim_{x \to 0^+} \frac{\log(x)}{(\tan(x))^{-1}}.
\]
Now we can use de L’Hospital’s rule:
\[
\lim_{x \to 0^+} \frac{\log(x)}{(\tan(x))^{-1}} = \lim_{x \to 0^+} \frac{x^{-1}}{-(\tan(x))^{-2} \sec^2(x)} = \lim_{x \to 0^+} -\frac{1}{x \cos^2(x) \frac{1}{\cos^2(x)}} = \lim_{x \to 0^+} -\frac{1}{x \sin^2(x)} = \lim_{x \to 0^+} -\frac{\sin^2(x)}{x} = \lim_{x \to 0^+} -\sin(x) \frac{\sin(x)}{x} = -0 \cdot 1 = 0.
\]