Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

Consider a function, \( h(x) \), whose third Maclaurin polynomial is \(-x + 2x^2 + \frac{2}{3}x^3\).

(a) What is \( h^{(3)}(0) \)?

**Answer:** 4

**Solution:** The third Maclaurin polynomial for \( h(x) \) is

\[
h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h^{(3)}(0)}{6} \cdot x^3 = -x + 2x^2 + \frac{2}{3}x^3.
\]

Thus \( h^{(3)}(0) = 6 \cdot \frac{2}{3} = 4 \) and for the next part, we note that \( h(0) = 0 \).

(b) What is \( h(0) \)?

**Answer:** 0

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Find the global maximum and the global minimum for \( f(x) = 2x^3 + 9x^2 + 2 \) on the interval \([-4, -1]\).

**Solution:** We compute \( f'(x) = 6x^2 + 18x \), which means that \( f(x) \) has no singular points (i.e., it is differentiable for all values of \( x \)), but it has two critical points obtained by solving \( f'(x) = 0 \), i.e. \( 6x(x - 3) = 0 \) which yields the two critical points \( x = 0 \) and \( x = 3 \). In order to compute the global maximum and the global minimum for \( f(x) \) on the interval \([-4, -1]\), we compute

\[
f(-4) = 18, \ f(-3) = 29 \text{ and } f(-1) = 9.
\]

So, the global maximum is \( f(-3) = 29 \) while the global minimum is \( f(-1) = 9 \).

(b) Consider a function \( f(x) \) which has \( f'''(x) = \frac{e^x}{4 - x} \). Show that when we approximate \( f(0) \) using its second Taylor polynomial around \( x = -1 \), the absolute error is less than \( \frac{1}{20} = 0.05 \).

**Solution:**

- The error is bounded (in absolute value) by

\[
\max_{c \in [-1,0]} \left| \frac{f'''(c)}{3!} \cdot (0 - (-1))^3 \right| = \max_{c \in [-1,0]} \left| \frac{e^c}{6(4-c)} \right|.
\]

- Since \( c \in [-1,0] \), we know that \( \left| \frac{e^c}{6(4-c)} \right| = \frac{e^c}{6(4-c)} \) since both numerator and denominator are positive.
• When $-1 \leq c \leq 0$, we know that $e^{-1} \leq e^c \leq e^0 = 1$ and $5 \leq 4-c \leq 4$, and that numerator and denominator are non-negative, so

$$\left| \frac{e^c}{6(4-c)} \right| = \frac{e^c}{6(4-c)} \leq \frac{1}{6(4-c)} \leq \frac{1}{6 \cdot 4} = \frac{1}{24} \leq \frac{1}{20}$$

as required.

• Alternatively, notice that $e^c$ is an increasing function of $c$, while $4-c$ is a decreasing function of $c$. Hence the fraction is an increasing function of $c$ and takes its largest value at $c = 0$. Hence

$$\left| \frac{e^c}{6(4-c)} \right| \leq \frac{1}{6 \times 4} = \frac{1}{24} \leq \frac{1}{20}.$$

**Marking scheme:**

• 1 mark for writing that the error is bounded (in absolute value) by

$$\max_{c \in [-1,0]} \left| \frac{f'''(c)}{3!} \cdot (0 - (-1))^3 \right| = \max_{c \in [-1,0]} \left| \frac{e^c}{6(4-c)} \right|.$$

• 1 mark for explaining why $c = 0$ is the right choice and then verifying that in that case the error is still bounded above by 0.05.

• The students lose 1 mark if they don’t explain why $c = 0$ is the right choice (and they simply plug in $c = 0$, or compare the values they get between plugging in $c = 0$ and $c = -1$).

**Long answer question — you must show your work**

3. [4 marks] A 20m long extension ladder leaning against a wall starts collapsing at a rate of 2m/s, while the foot of the ladder remains a constant 5m from the wall. How fast is the ladder moving down the wall after 3.5 seconds?

**Solution:**

• If we write $z(t)$ for the length of the ladder at time $t$ and $y(t)$ for the height of the top end of the ladder at time $t$ we have

$$z(t)^2 = 5^2 + y(t)^2.$$

• We differentiate the above equation with respect to $t$ and get

$$2z \cdot z' = 2y \cdot y'.$$
• We are told that $z'(2.5) = -2$ and $z(2.5) = 20 - 3.5 \cdot 2 = 13$.

• At this point $y = \sqrt{z^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$.

• Hence

\[
2 \cdot 13 \cdot (-2) = 2 \cdot 12y' \\
y' = -\frac{2 \cdot 13}{12} = -\frac{13}{6} \text{ meters per second.}
\]

**Marking scheme:**

• 1 mark for obtaining the equation $2z(t) \cdot z'(t) = 2x \cdot x'$.

• 1 mark for $z' = -2$, $z = 13$ all correct.

• 1 mark for computing $y(2.5) = 12$.

• 1 mark for obtaining the correct answer $y'(2.5) = -\frac{13}{6} \text{ m/s.}$