Very short answer questions

1. **2 marks** Each part is worth 1 mark. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise
   
   (a) Compute
   \[
   \lim_{x \to +\infty} \frac{x^3 + 2x^2 - 1}{4x^3 + 3x + 5}.
   \]
   Answer: \(\frac{1}{4}\)
   
   **Solution:** We have, after dividing both numerator and denominator by \(x^3\) (which is the highest power of the denominator) that
   \[
   \frac{x^3 + 2x^2 - 1}{4x^3 + 3x + 5} = \frac{1 + \frac{2}{x} - \frac{1}{x^3}}{4 + \frac{3}{x} + \frac{5}{x^3}}.
   \]
   Since \(1/x^n \to 0\) as \(x \to +\infty\), we conclude that
   \[
   \lim_{x \to +\infty} \frac{x^3 + 2x^2 - 1}{4x^3 + 3x + 5} = \frac{1}{4}.
   \]

   (b) Compute the derivative of \(\left(\frac{x - 2}{3x^2 + x}\right)\)
   Answer: \(-\frac{3x^2 + 12x + 2}{(3x^2 + x)^2}\)
   
   **Solution:** We use the quotient rule:
   \[
   \frac{1 \cdot (3x^2 + x) - (x - 2)(6x + 1)}{(3x^2 + x)^2} = -\frac{3x^2 + 12x + 2}{(3x^2 + x)^2}.
   \]

Short answer questions — you must show your work

2. **4 marks** Each part is worth 2 marks.
   
   (a) Evaluate
   \[
   \lim_{x \to -\infty} \frac{8x - 5}{\sqrt{4x^2 + x} - 6}.
   \]
   
   **Solution:** We divide by the highest power of the denominator, which is \(x\) and note that (for \(x < 0\))
   \[
   \frac{\sqrt{4x^2 + x}}{x} = -\sqrt{\frac{4x^2 + x}{x^2}} = -\sqrt{4 + \frac{1}{x}}.
   \]
   Since \(1/x \to 0\) as \(x \to -\infty\), we conclude that
   \[
   \lim_{x \to -\infty} \frac{8x - 5}{\sqrt{4x^2 + x} - 6} = \lim_{x \to -\infty} \frac{8 - \frac{5}{x}}{-\sqrt{4 + \frac{1}{x} - \frac{6}{x}}} = \frac{8}{-2} = -4.
   \]
Marking scheme:

- 1 mark for realizing that $\sqrt{\frac{4x^2 + x}{x}} = -\sqrt{4 + \frac{1}{x}}$.
- 1 mark for correct answer.

(b) Find the equation of the tangent line to the graph of $y = x^3 - 2x^2 - 1$ at $x = 2$.

Solution: We compute the derivative of $x^3 - 2x^2 - 1$ as being $3x^2 - 4x$, which evaluated at $x = 2$ yields 4. Since we also compute $2^3 - 2 \cdot 2^2 - 1 = -1$, then the equation of the tangent line is

$$y + 1 = 4(x - 2) \quad \text{or} \quad y = 4x - 9.$$ 

Marking scheme:

- 1 mark for computing correctly the slope of the tangent.
- 1 mark for correct equation of the tangent line (in either form).

Long answer question — you must show your work

3. 4 marks Show that there exists at least one real number $c$ such that $2 \cos(\frac{x}{2}) = \sin(x) - \frac{1}{x}$.

Solution: We let $f(x) = 2 \cos(\frac{x}{2}) - \sin(x) + \frac{1}{x}$. Then $f(x)$ is continuous on the positive real numbers since $\cos(x)$ and $\sin(x)$ are everywhere continuous and $\frac{1}{x}$ is continuous everywhere except at $x = 0$.

We find a positive value $a$ such that $f(a) > 0$. We observe that $a = \pi$ works since

$$f(\pi) = 2 \cos(\frac{\pi}{2}) - \sin(\pi) + \frac{1}{\pi} = 0 + 0 + \frac{1}{\pi} > 0.$$ 

We find a positive value $b$ such that $f(b) < 0$. We see that $b = 2\pi$ works since

$$f(2\pi) = 2 \cos(\pi) - \sin(2\pi) + \frac{1}{2\pi} = -2 + 0 + \frac{1}{2\pi} < 0,$$

because $\frac{1}{2\pi} < 2$.

So, because $f(x)$ is continuous on $[\pi, 2\pi]$ and $f(\pi) < 0$ while $f(2\pi) > 0$, then the Intermediate Value Theorem guarantees the existence of a real number $c \in (\pi, 2\pi)$ such that $f(c) = 0$. 

Marking scheme:

- 1 mark for constructing a function $f(x)$ as a difference of the two given functions and for writing that $f(x)$ is a continuous function on a correct interval $I$ (i.e. one that does not include 0)
• 1 mark for finding a value $a \in I$ such that $f(a) > 0$.
• 1 mark for finding a value $b \in I$ such that $f(b) < 0$.
• 1 mark for the correct conclusion which should mention that the solution $c$ is in between $a$ and $b$ and its existence is justified by the Intermediate Value Theorem.
• If $a$ and $b$ are on different sides of 0, give at most 2 points total.