Very short answer questions

1. \[ \frac{2 \text{ marks}}{} \] Each part is worth 1 marks. Please write your answers in the boxes. 

**Marking scheme:** 1 for each correct, 0 otherwise

Consider the function, \( h(x) = 2x^3 - 6x^2 + 2 \).

(a) What are the coordinates of the local maximum of \( h(x) \)?

**Answer:** \((0, 2)\)

**Solution:** The function \( h(x) \) has critical points, but no singular points since its derivative \( h'(x) = 6x^2 - 12x \) is defined for all values of \( x \). The critical points of \( h(x) \) are computed by equating \( h(x) = 0 \) which yields

\[ 6x^2 - 12x = 0, \text{ i.e. } x(x - 2) = 0, \]

and thus \( x = 0 \) and \( x = 2 \). Using either the Second Derivative Test, i.e. computing \( h''(x) = 12x - 12 \) and then plugging in the critical numbers \( x = 0 \) and \( x = 2 \) in \( h''(x) \), or by simply noticing that \( h'(x) \) changes from positive to negative at \( x = 0 \) and from negative to positive at \( x = 2 \), we conclude that \( x = 0 \) is a point of local maximum, while \( x = 2 \) is a point of local minimum. We compute \( f(0) = 2 \) and \( f(2) = 2 \times 8 - 6 \times 4 + 2 = -6 \).

(b) What are the coordinates of the local minimum of \( h(x) \)?

**Answer:** \((2, -6)\)

Short answer questions — you must show your work

2. \[ \frac{4 \text{ marks}}{} \] Each part is worth 2 marks.

(a) Find the intervals where \( f(x) = \arcsin(x) + 2\sqrt{1-x^2} \) is increasing.

**Solution:** First of all, \( f(x) \) is only defined on \([-1, 1]\). In order to find where \( f(x) \) is increasing we compute \( f'(x) \) and see where it is positive.

\[ f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{2x}{\sqrt{1-x^2}} = \frac{1 - 2x}{\sqrt{1-x^2}} \]

This is defined on \((-1, 1)\). The denominator is positive on that domain, while the numerator is positive when \( 1 - 2x > 0 \). Hence \( f \) is increasing on \((-1, 1/2)\).

**Marking scheme:**

- 1 mark for computing \( f'(x) \), i.e. \( f'(x) = \frac{1 - 2x}{\sqrt{1-x^2}} \).
- 1 mark for writing the correct interval where \( f(x) \) is increasing; also accept interval including one or both endpoints.
(b) Let \( f(x) = (x - \pi)^2 - \sin(x) + \cos(x) \). Show that there exists a real number \( c \) such that \( f'(c) = 0 \).

**Solution:** We note that \( f(0) = f(2\pi) = \pi^2 - 0 + 1 \). Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists \( c \in (0, 2\pi) \) such that

\[
f'(c) = \frac{f(2\pi) - f(0)}{2\pi - 0} = 0.
\]

**Marking scheme:**

- 1 mark for writing that \( f(0) = f(2\pi) = 1 + \pi^2 \).
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of \( c \in (0, 2\pi) \) such that \( f'(c) = 0 \).
- Alternatively the students could differentiate \( f \) and then apply Intermediate Value Theorem (IVT) for \( f'(x) \). Since \( f'(x) = 2(x - \pi) - \cos x - \sin x \), one notes that \( f(0) = -2\pi - 1 < 0 \) and \( f(2\pi) = 2\pi - 1 > 0 \) and since \( f'(x) \) is continuous for all real values, then there exists \( c \in (0, 2\pi) \) such that \( f'(c) = 0 \), by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

**Long answer question — you must show your work**

3. **4 marks** Find the global maximum and the global minimum for \( f(x) = 3x^4 - 4x^3 + 3 \) on the interval \([-1, 2]\).

**Solution:** We compute \( f'(x) = 12x^3 - 12x^2 \), which means that \( f(x) \) has no singular points (i.e., it is differentiable for all values of \( x \)), but it has two critical points obtained by solving \( f'(x) = 0 \), i.e., \( 0 = 12(x^3 - x^2) = 4x^2(x - 1) \) which yields the two critical points \( x = 0 \) and \( x = 1 \). In order to compute the global maximum and the global minimum for \( f(x) \) on the interval \([-1, 2]\), we compute

\[
\begin{align*}
f(-1) &= 3 + 4 + 3 = 10 \\
f(0) &= 3 \\
f(1) &= 2 \\
f(2) &= 48 - 32 + 3 = 19
\end{align*}
\]

So, the global maximum is \( f(2) = 19 \) while the global minimum is \( f(1) = 2 \).

**Marking scheme:**

- 1 mark for finding \( x = 0, 1 \) as the only critical (and singular) points in \([-1, 2]\).
- 2 marks for computing \( f(-1), f(2), f(0) \) and \( f(1) \). If they compute one value wrong (or they are not computed), then they lose 1 mark. If at least two values values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for writing correctly that \( f(1) = 2 \) is the global minimum, while \( f(2) = 19 \) is the global maximum.
Very short answer questions

1. \(2\) marks Each part is worth 1 mark. Please write your answers in the boxes. \textbf{Marking scheme:} 1 for each correct, 0 otherwise

Consider the function, \(h(x) = x^3 - 3x + 5\).

(a) What are the coordinates of the \underline{local} maximum of \(h(x)\)?

\textbf{Answer:} \((-1, 7)\)

\textbf{Solution:} The function \(h(x)\) has critical points, but no singular points since its derivative \(h'(x) = 3x^2 - 3\) is defined for all values of \(x\). The critical points of \(h(x)\) are computed by equating \(h(x) = 0\) which yields

\[3x^2 - 3 = 0, \text{ i.e. } x^2 = 1,\]

and thus \(x = -1\) and \(x = 1\). Using either the Second Derivative Test, i.e. computing \(h''(x) = 6x\) and then plugging in the critical numbers \(x = -1\) and \(x = 1\) in \(h''(x)\), or by simply noticing that \(h'(x)\) changes from positive to negative at \(x = -1\) and from negative to positive at \(x = 1\), we conclude that \(x = -1\) is a point of local maximum, while \(x = 1\) is a point of local minimum. We compute \(f(-1) = 7\) and \(f(1) = 3\).

(b) What are the coordinates of the \underline{local} minimum of \(h(x)\)?

\textbf{Answer:} \((1, 3)\)

Short answer questions — you must show your work

2. \(4\) marks Each part is worth 2 marks.

(a) Find the intervals where \(f(x) = \sqrt{x + 6}\) is increasing.

\textbf{Solution:} First of all, \(f(x)\) is defined for all \(x \geq 0\) due to the presence of the square-root. In order to find where is \(f(x)\) increasing, we find where is \(f'(x)\) positive. So,

\[f'(x) = \frac{1}{2\sqrt{x + 6}} \cdot (x + 6) - \frac{\sqrt{x} \cdot 1}{(x + 6)^2} = \frac{x + 6 - 2x}{2\sqrt{x} \cdot (x + 6)^2} = \frac{6 - x}{2\sqrt{x} \cdot (x + 6)^2}\]

and thus, since the denominator is always positive, we conclude that \(f(x)\) is increasing when \(f'(x) > 0\), i.e. when \(6 - x > 0\). Recalling that the domain of definition for \(f(x)\) is \([0, +\infty)\), we conclude that \(f(x)\) is increasing on the interval \((0, 6)\). \textbf{Marking scheme:}

- 1 mark for computing \(f'(x)\), i.e. \(f'(x) = \frac{6-x}{2\sqrt{x}(x+6)^2}\).
- 1 mark for writing the correct interval where \(f(x)\) is increasing; it is acceptable also \([0, 6]\), but \textbf{not} acceptable \((-\infty, 6)\).
(b) Let \( f(x) = x^2 - 2\pi x - \sin(x) \). Show that there exists a real number \( c \) such that \( f'(c) = 0 \).

**Solution:** We note that \( f(0) = f(2\pi) = 0 \). Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists \( c \in (0, 2\pi) \) such that

\[
f'(c) = \frac{f(2\pi) - f(0)}{2\pi - 0} = 0.
\]

**Marking scheme:**

- 1 mark for writing that \( f(0) = f(2\pi) = 0 \).
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of \( c \in (0, 2\pi) \) such that \( f'(c) = 0 \).
- Alternatively the students could differentiate \( f \) and then apply Intermediate Value Theorem (IVT) for \( f'(x) \). Since \( f'(x) = 2x - 2\pi - \cos(x) \), one notes that \( f'(0) = -2\pi - 1 < 0 \) and \( f'(2\pi) = 2\pi - 1 > 0 \) and since \( f'(x) \) is continuous for all real values, then there exists \( c \in (0, 2\pi) \) such that \( f'(c) = 0 \), by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

**Long answer question — you must show your work**

3. [4 marks] Find the global maximum and the global minimum for \( f(x) = x^3 - 6x^2 + 2 \) on the interval \([3, 5]\).

**Solution:** We compute \( f'(x) = 3x^2 - 12x \), which means that \( f(x) \) has no singular points (i.e., it is differentiable for all values of \( x \)), but it has two critical points obtained by solving \( f'(x) = 0 \), i.e. \( 3x(x - 4) = 0 \) which yields the two critical points \( x = 0 \) and \( x = 4 \). In order to compute the global maximum and the global minimum for \( f(x) \) on the interval \([3, 5]\), we compute

\[
f(3) = -25, f(4) = -30 \text{ and } f(5) = -23.
\]

So, the global maximum is \( f(5) = -23 \) while the global minimum is \( f(4) = -30 \).

**Marking scheme:**

- 1 mark for writing that \( x = 4 \) is the only critical (and singular) point of \( f(x) \) in the interval \([3, 5]\).
- 2 marks for computing \( f(3) \), \( f(4) \) and \( f(5) \). If they compute one or two values wrong (or they are not computed), then they lose 1 mark. If all three values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for stating BOTH \( f(4) = -30 \) is the global minimum, and \( f(5) = -23 \) is the global maximum.
Very short answer questions

1. 2 marks Each part is worth 1 marks. Please write your answers in the boxes. Marking scheme: 1 for each correct, 0 otherwise
Consider the function, \( h(x) = x^3 - 12x + 4 \).
(a) What are the coordinates of the local maximum of \( h(x) \)?

**Answer:** \((-2, 20)\)

**Solution:** The function \( h(x) \) has critical points, but no singular points since its derivative \( h'(x) = 3x^2 - 12 \) is defined for all values of \( x \). The critical points of \( h(x) \) are computed by equating \( h(x) = 0 \) which yields
\[
3x^2 - 12 = 0, \text{ i.e. } x^2 = 4,
\]
and thus \( x = -2 \) and \( x = 2 \). Using either the Second Derivative Test, i.e. computing \( h''(x) = 6x \) and then plugging in the critical numbers \( x = -2 \) and \( x = 2 \) in \( h''(x) \), or by simply noticing that \( h'(x) \) changes from positive to negative at \( x = -2 \) and from negative to positive at \( x = 2 \), we conclude that \( x = -2 \) is a point of local maximum, while \( x = 2 \) is a point of local minimum. We compute \( f(-2) = 20 \) and \( f(2) = -12 \).

(b) What are the coordinates of the local minimum of \( h(x) \)?

**Answer:** \((2, -12)\)

Short answer questions — you must show your work

2. 4 marks Each part is worth 2 marks.
(a) Find the intervals where \( f(x) = \sqrt{\frac{x-1}{2x+4}} \) is increasing.

**Solution:** First of all, \( f(x) \) is defined for all \( x \geq 1 \) due to the presence of the square-root. We also need \( x \neq -2 \) for the denominator, but this is covered by \( x \geq 1 \). In order to find where is \( f(x) \) increasing, we find where is \( f'(x) \) positive. So,
\[
f'(x) = \frac{2x+4}{2\sqrt{x-1}} - 2\sqrt{x-1} \frac{2x+4}{(2x+4)^2} = \frac{(x+2) - 2(x-1)}{\sqrt{x-1}(2x+4)^2} = \frac{-x+4}{\sqrt{x-1}(2x+4)^2}
\]
Note the denominator is never negative, so we conclude that \( f(x) \) is increasing when the numerator of \( f'(x) \) is positive, i.e. when \( 4 - x > 0 \), or \( x < 4 \). Recalling that the domain of definition for \( f(x) \) is \([1, +\infty)\), we conclude that \( f(x) \) is increasing on the interval \((1, 4)\). **Marking scheme:**

- 1 mark for computing \( f'(x) \), i.e. \( f'(x) = \frac{4-x}{\sqrt{x-1}(2x+4)^2} \).
- 1 mark for writing the correct interval where \( f(x) \) is increasing; it is acceptable also \([1, 4]\), but **not** acceptable \((-\infty, 4)\).
(b) Let $f(x) = x^2 - 3\pi x + \sin(x)$. Show that there exists a real number $c$ such that $f'(c) = 0$.

Solution: We note that $f(0) = f(3\pi) = 0$. Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists $c \in (0, 3\pi)$ such that

$$f'(c) = \frac{f(3\pi) - f(0)}{3\pi - 0} = 0.$$ 

Marking scheme:

- 1 mark for writing that $f(0) = f(3\pi) = 0$.
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of $c \in (0, 3\pi)$ such that $f'(c) = 0$.
- Alternatively the students could differentiate $f$ and then apply Intermediate Value Theorem (IVT) for $f'(x)$. Since $f'(x) = 2x - 3\pi + \cos(x)$, one notes that $f'(0) = -3\pi + 1 < 0$ and $f'(3\pi) = 3\pi - 1 > 0$ and since $f'(x)$ is continuous for all real values, then there exists $c \in (0, 3\pi)$ such that $f'(c) = 0$, by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

Long answer question — you must show your work

3. [4 marks] Find the global maximum and the global minimum for $f(x) = x^5 - 5x - 10$ on the interval $[0, 2]$.

Solution: We compute $f'(x) = 5x^4 - 5$, which means that $f(x)$ has no singular points (i.e., it is differentiable for all values of $x$), but it has two critical points obtained by solving $f'(x) = 0$, i.e. $5(x^4 - 1) = 0$ which yields the two critical points $x = -1$ and $x = 1$. In order to compute the global maximum and the global minimum for $f(x)$ on the interval $[0, 2]$, we compute

$$f(0) = -10, \ f(1) = -14 \text{ and } f(2) = 12.$$ 

So, the global maximum is $f(2) = 12$ while the global minimum is $f(1) = -14$.

Marking scheme:

- 1 mark for writing that $x = 1$ is the only critical (and singular) point of $f(x)$ in the interval $[0, 2]$. 
- 2 marks for computing $f(0), f(1)$ and $f(2)$. If they compute one or two values wrong (or they are not computed), then they lose 1 mark. If all three values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for stating BOTH $f(1) = -14$ is the global minimum, and $f(2) = 12$ is the global maximum.
Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

Consider the function, \( h(x) = 2x^3 - 6x + 2 \).

(a) What are the coordinates of the **local** maximum of \( h(x) \)?

**Answer:** \((-1, 6)\)

**Solution:** The function \( h(x) \) has critical points, but no singular points since its derivative \( h'(x) = 6x^2 - 6 \) is defined for all values of \( x \). The critical points of \( h(x) \) are computed by equating \( h(x) = 0 \) which yields

\[ 6x^2 - 6 = 0, \text{ i.e. } x^2 - 1 = 0, \]

and thus \( x = -1 \) and \( x = +1 \). Using either the Second Derivative Test, i.e. computing \( h''(x) = 6x \) and then plugging in the critical numbers \( x = -1 \) and \( x = +1 \) in \( h''(x) \), or by simply noticing that \( h'(x) \) changes from positive to negative at \( x = -1 \) and from negative to positive at \( x = +1 \), we conclude that \( x = -1 \) is a point of local maximum, while \( x = +1 \) is a point of local minimum. We compute \( f(1) = -2 \) and \( f(-1) = -2 + 6 + 2 = 6 \).

(b) What are the coordinates of the **local** minimum of \( h(x) \)?

**Answer:** \((1, -2)\)

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Find the intervals where \( f(x) = xe^{-x^2/2} \) is increasing.

**Solution:** The function is defined on all reals. Its derivative is

\[ f'(x) = e^{-x^2/2} + x \cdot (-x) \cdot e^{-x^2/2} = (1 - x^2)e^{-x^2/2} \]

Since \( e^{\text{blah}} > 0 \), the sign of the derivative is determined by the sign of \((1 - x^2)\). Hence the function is increasing when \((1 - x^2) > 0\), that is when \( x \in (-1, 1) \).

**Marking scheme:**

- 1 mark for computing \( f'(x) \), i.e. \( f'(x) = (1 - x^2)e^{-x^2/2} \).
- 1 mark for writing the correct interval where \( f(x) \) is increasing; also accept interval with one or both endpoints.

(b) Let \( f(x) = (x + \pi)^2 + \cos(x) \). Show that there exists a real number \( c \) such that \( f'(c) = 0 \).
Solution: We note that \( f(0) = f(-2\pi) = \pi^2 + 1 \). Then using the Mean Value Theorem (note that the function is differentiable for all real numbers), we get that there exists \( c \in (-2\pi, 0) \) such that

\[
    f'(c) = \frac{f(-2\pi) - f(0)}{-2\pi} = 0.
\]

Marking scheme:

- 1 mark for writing that \( f(0) = f(-2\pi) = 1 + \pi^2 \).
- 1 mark for invoking Mean Value Theorem (or Rolle) correctly to conclude the existence of \( c \in (-2\pi, 0) \) such that \( f'(c) = 0 \).
- Alternatively the students could differentiate \( f \) and then apply Intermediate Value Theorem (IVT) for \( f'(x) \). Since \( f'(x) = 2(x + \pi) - \sin x \), one notes that \( f(0) = 2\pi > 0 \) and \( f(-2\pi) = -2\pi < 0 \) and since \( f'(x) \) is continuous for all real values, then there exists \( c \in (-2\pi, 0) \) such that \( f'(c) = 0 \), by IVT. For this approach = 1 mark for evaluating derivative at 2 sensible points and 1 mark for applying IVT.

Long answer question — you must show your work

3. **4 marks** Find the global maximum and the global minimum for \( f(x) = 4x^3 - 6x^2 + 3 \) on the interval \([-1, 2]\).

Solution: We compute \( f'(x) = 12x^2 - 12x \), which means that \( f(x) \) has no singular points (i.e., it is differentiable for all values of \( x \)), but it has two critical points obtained by solving \( f'(x) = 0 \), i.e. \( 0 = 12x(x - 1) \) which yields the two critical points \( x = 0 \) and \( x = 1 \). In order to compute the global maximum and the global minimum for \( f(x) \) on the interval \([-1, 2]\), we compute

\[
    f(-1) = -4 - 6 + 3 = -7 \\
    f(0) = 3 \\
    f(1) = 1 \\
    f(2) = 32 - 24 + 3 = 11
\]

So, the global maximum is \( f(2) = 11 \) while the global minimum is \( f(-1) = -7 \).

Marking scheme:

- 1 mark for finding \( x = 0, 1 \) as the only critical (and singular) points in \([-1, 2]\).
- 2 marks for computing \( f(-1), f(2), f(0) \) and \( f(1) \). If they compute one value wrong (or they are not computed), then they lose 1 mark. If at least two values values are computed wrong (or not computed), then they lose 2 marks.
- 1 mark for writing correctly that \( f(2) = 11 \) is the global max, while \( f(-1) = -7 \) is the global maximum.