Very short answer questions

1. Each part is worth 1 mark. Please write your answers in the boxes. **Marking scheme**: 1 for each correct, 0 otherwise

Consider a function, \( h(x) \), whose third-degree Maclaurin polynomial is \( 1 - 3x + \frac{1}{6}x^2 + \frac{2}{7}x^3 \).

(a) What is \( h'(0) \)?

**Answer**: \(-3\)

**Solution**: The third Maclaurin polynomial for \( h(x) \) is

\[
h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h'''(0)}{6} \cdot x^3 = 1 - 3x + \frac{1}{6}x^2 + \frac{2}{7}x^3
\]

Thus \( h'(0) = -3 \) and for the next part, we note that \( h''(0) = \frac{1}{3} \).

(b) What is \( h''(0) \)?

**Answer**: \( \frac{1}{3} \)

Short answer questions — you must show your work

2. Each part is worth 2 marks.

(a) Estimate \( \sqrt[4]{15} \) using a linear approximation

**Solution**: We use the function \( f(x) = x^{1/4} \) and point \( a = 15 \) as the centre of our approximation since we know that

\[
f'(x) = \frac{1}{4}x^{-3/4}; \quad f'(16) = \frac{1}{4} \cdot \frac{1}{16} = \frac{1}{32}.
\]

So, a linear approximation of \( \sqrt[4]{15} = f(15) \) is

\[
T_1(15) = f(16) + f'(16) \cdot (15 - 16) = 2 - \frac{1}{32} = \frac{63}{32}
\]

**Marking scheme**:

- 1 mark for writing that the function to use for the linear approximation is \( f(x) = \sqrt[4]{x} \), that \( a = 16 \). Also accept nearby \( a \) values that make it easy to compute \( f(a), f'(a) \).
- 1 mark for obtaining the linear approximation \( 63/32 \). (or equivalent if they use other acceptable \( a \))
(b) Consider a function $f(x)$ which has $f^{(3)}(x) = \frac{x}{10 - \sin x}$. Show that when we approximate $f(1)$ using its second degree Maclaurin polynomial, the absolute value of the error is less than $\frac{1}{50} = 0.02$.

**Solution:**

- The error is bounded (in absolute value) by
  \[
  \max_{c \in [0,1]} \left| \frac{f'''(c)}{3!} \cdot (1 - 0)^3 \right| = \max_{c \in [0,1]} \left| \frac{c}{6(10 - \sin c)} \right|.
  \]

- When $0 \leq c \leq 1$, we know that $-1 \leq \sin c \leq 1$. Hence $9 \leq 10 - \sin c \leq 11$.

- Hence
  \[
  \left| \frac{c}{6(10 - \sin c)} \right| = \frac{|c|}{6|10 - \sin c|} \leq \frac{1}{6|10 - \sin c|} \leq \frac{1}{6 \times 9} = \frac{1}{54} < \frac{1}{50}.
  \]

**Marking scheme:**

- 1 mark for writing that the error is bounded (in absolute value) by
  \[
  \max_{c \in [0,1]} \left| \frac{f'''(c)}{3!} \cdot (1 - 0)^3 \right| = \max_{c \in [0,1]} \left| \frac{c}{10 - \sin c} \right|.
  \]
  (or similar statement)

- 1 mark for explaining their error bound (provided the error is still bounded above by 0.02). Be reasonably generous.

- The students lose 1 mark if they get the right bound, but don’t explain it.

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**Long answer question — you must show your work**

3. **4 marks** Two particles move in the cartesian plane. Particle $A$ starts at $(3, 0)$ while particle $B$ starts at $(0, 0)$. Particle $A$ moves in the $+y$ direction at 1 unit per second, while $B$ moves in the $-y$ direction at 3 units per second. How fast is the distance between the particles changing when the distance between them is 5 units?

**Solution:**

- Let $y_1$ be the distance travelled by $A$ and $y_2$ be the distance travelled by $B$. 
• The distance between them is then
\[ z^2 = 9 + (y_1 + y_2)^2 \]

• We differentiate the above equation with respect to \( t \) and get
\[ 2z \cdot z' = 2(y_1 + y_2) \cdot (y_1' + y_2') \]

• We need to determine \( y_1, y_2 \) when \( z = 5 \). We have
\[ 5^2 = 9 + (y_1 + y_2)^2 \]

Hence \( y_1 + y_2 = 4 \). So we must have \( y_1 = 1, y_2 = 3 \).

• We are told that \( y_1' = 1 \) and \( y_2' = 3 \). Hence
\[ 2 \times 5 \times z' = 2 \times 4 \times 4 \]
\[ z' = \frac{16}{5} \text{ units / second} \]

• Equivalently we can let \( y \) be the vertical distance between the particles. Hence \( y' = 4 \) and
\[ z^2 = 9 + y^2 \]
\[ 2zz' = 2yy' \]

When \( z = 5, y = 4 \) and so
\[ 10z' = 32 \]

Thus \( z' = \frac{16}{5} \) units / second.

**Marking scheme:**

• 1 mark for \( z^2 = 9 + (y_1 + y_2)^2 \) or equivalently \( z = 9 + y^2 \).

• 1 mark for obtaining the equation \( 2z(t) \cdot z'(t) = 2(y_1 + y_2)(y_1' + y_2') \) or equivalently \( 2zz' = 2yy' \).

• 1 mark for \( y_1 = 1, y_2 = 3 \) all correct (equivalently \( y = 4 \)) when \( z = 5 \).

• 1 mark for obtaining the correct answer \( z' = \frac{16}{5} \).
Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

Consider a function, \( h(x) \), whose third-degree Maclaurin polynomial is \( 5 - \frac{1}{3}x^2 + 2x^3 \).

(a) What is \( h'(0) \)?

**Answer:** 0

**Solution:** The third Maclaurin polynomial for \( h(x) \) is

\[
h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h'''(0)}{6} \cdot x^3 = 5 - \frac{x^2}{3} + 2x^3.
\]

Thus \( h'(0) = 0 \) and for the next part, we note that \( h''(0) = -\frac{2}{3} \).

(b) What is \( h''(0) \)?

**Answer:** \(-\frac{2}{3}\)

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Estimate \( \sqrt{35} \) using a linear approximation

**Solution:** We use the function \( f(x) = \sqrt{x} \) and point \( a = 36 \) as the centre of our approximation since we know that

\[
f(a) = f(36) = \sqrt{36} = 6.
\]

We compute \( f'(x) = \frac{1}{2\sqrt{x}} \); so

\[
f'(36) = \frac{1}{2\sqrt{36}} = \frac{1}{12}.
\]

So, a linear approximation of \( \sqrt{35} = f(35) \) is

\[
T_1(35) = f(36) + f'(36) \cdot (35 - 36) = 6 - \frac{1}{12} = \frac{71}{12}.
\]

**Marking scheme:**

- 1 mark for writing that the function to use for the linear approximation is \( f(x) = \sqrt{x} \), that \( a = 36 \). Also accept nearby \( a \) values that make it easy to compute \( f(a), f'(a) \) (eg 25,49).

- 1 mark for obtaining the linear approximation \( 6 - \frac{1}{12} \). (or equivalent if they use other acceptable \( a \))
(b) Consider a function \( f(x) \) which has \( f^{(3)}(x) = \frac{x^3}{10 - x^2} \). Show that when we approximate \( f(1) \) using its second degree Maclaurin polynomial, the absolute value of the error is less than \( \frac{1}{50} = 0.02 \).

Solution:

- The error is bounded (in absolute value) by

\[
\max_{c \in [0,1]} \left| \frac{f^{(3)}(c)}{3!} \cdot (1-0)^3 \right| = \max_{c \in [0,1]} \left| \frac{c^3}{6(10-c^2)} \right|.
\]

- Since \( c \in [0,1] \), we know that \( \left| \frac{c^3}{6(10-c^2)} \right| = \frac{c^3}{6(10-c^2)} \) since both numerator and denominator are positive.

- When \( 0 \leq c \leq 1 \), we know that \( 0 \leq c^3 \leq 1 \) and \( 9 \leq 10 - c^2 \leq 10 \), and that numerator and denominator are non-negative, so

\[
\left| \frac{c^3}{6(10-c^2)} \right| = \frac{c^3}{6(10-c^2)} \leq \frac{1}{6(10-c^2)} \leq \frac{1}{6 \cdot 9} = \frac{1}{54} \leq \frac{1}{50}
\]

as required.

- Alternatively, notice that \( c^3 \) is an increasing function of \( c \), while \( 10 - c^2 \) is a decreasing function of \( c \). Hence the fraction is an increasing function of \( c \) and takes its largest value at \( c = 1 \). Hence

\[
\left| \frac{c^3}{6(10-c^2)} \right| \leq \frac{1}{6 \cdot 9} = \frac{1}{54} \leq \frac{1}{50}.
\]

Marking scheme:

- 1 mark for writing that the error is bounded (in absolute value) by

\[
\max_{c \in [0,1]} \left| \frac{f^{(3)}(c)}{3!} \cdot (1-0)^3 \right| = \max_{c \in [0,1]} \left| \frac{c^3}{6(10-c^2)} \right|.
\]

(or similar statement)

- 1 mark for explaining why \( c = 1 \) is the right choice and then verifying that in that case the error is still bounded above by 0.02. Be reasonably generous.

- The students lose 1 mark if they don’t explain why \( c = 1 \) is the right choice (and simply they plug in \( c = 1 \), or compare the values they get between plugging in \( c = 0 \) and \( c = 1 \)).
Two particles move in the cartesian plane. Particle A travels on the $x$-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the $y$-axis starting at $(0, 12)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

**Solution:**

- We compute the distance $z(t)$ between the particle and the point with coordinates $(0, 3)$ at each moment in time as
  
  $$z^2(t) = x(t)^2 + y(t)^2,$$

  where $x(t)$ is the position on the $x$-axis of the particle A at time $t$ (measured in seconds) and $y(t)$ is the position on the $y$-axis of the particle B at the same time $t$.

- We differentiate the above equation with respect to $t$ and get
  
  $$2z \cdot z' = 2x \cdot x' + 2y \cdot y',$$

- We are told that $x' = -2$ and $y' = -3$. Further it will take 3 seconds for particle A to reach $x = 4$, and in this time particle B will reach $y = 3$.

- Alternatively (equivalently??) write $x = 10 - 2t, y = 12 - 3t$ to get $t = 3, x' = -2, y' = -3, y = 3$.

- At this point $z = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$.

- Hence
  
  $$10z' = 8 \cdot (-2) + 6 \cdot (-3) = -34$$

  $$z' = -\frac{34}{10} = -\frac{17}{5} \text{ units per second.}$$

**Marking scheme:**

- 1 mark for obtaining the equation $2z(t) \cdot z'(t) = 2x \cdot x' + 2y \cdot y'$.

- 1 mark for $x' = -2, y' = -3, y = 3$ all correct.

- 1 mark for computing $z(3) = 5$.

- 1 mark for obtaining the correct answer $z'(3) = -\frac{17}{5}$. 
Very short answer questions

1. [2 marks] Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme:** 1 for each correct, 0 otherwise

Consider a function, \( f(x) \), whose third-degree Maclaurin polynomial is \( 4 + 3x^2 + \frac{1}{2}x^3 \).

(a) What is \( f'(0) \)?

**Answer:** 0

**Solution:** The third Maclaurin polynomial for \( f(x) \) is

\[
 f(0) + f'(0)x + \frac{f''(0)}{2} \cdot x^2 + \frac{f'''(0)}{6} \cdot x^3 = 4 + 3x^2 + \frac{1}{2}x^3.
\]

Thus \( f'(0) = 0 \) and for the next part, we note that \( f''(0) = 6 \).

(b) What is \( f''(0) \)?

**Answer:** 6

Short answer questions — you must show your work

2. [4 marks] Each part is worth 2 marks.

(a) Estimate \( \sqrt[3]{9} \) using a linear approximation

**Solution:** We use the function \( f(x) = \sqrt[3]{x} \) and point \( a = 8 \) as the centre of our approximation since we know that

\[
 f(a) = f(8) = \sqrt[3]{8} = 2.
\]

We compute \( f'(x) = \frac{1}{3 \sqrt[3]{x^2}} \); so

\[
 f'(8) = \frac{1}{3 \sqrt[3]{8^2}} = \frac{1}{12}.
\]

So, a linear approximation of \( \sqrt[3]{9} = f(9) \) is

\[
 T_1(9) = f(8) + f'(8) \cdot (9 - 8) = 2 + \frac{1}{12}.
\]

**Marking scheme:**

- 1 mark for writing that the function to use for the linear approximation is \( f(x) = \sqrt[3]{x} \), that \( a = 8 \). Also accept nearby \( a \) values that make it easy to compute \( f(a), f'(a) \).

- 1 mark for obtaining the linear approximation \( 2 + \frac{1}{12} \). (or equivalent if they use other acceptable \( a \))
(b) Consider a function \( f(x) \) which has \( f^{(3)}(x) = \frac{1}{5}e^{-2x} \cdot \sin(x) \). Show that when we approximate \( f(1) \) using its second degree Maclaurin polynomial, the absolute value of the error is less than \( \frac{1}{30} \).

**Solution:**

- The error is bounded (in absolute value) by
  \[
  \max_{c \in [0,1]} \left| \frac{f'''(c)}{3!} \cdot (1 - 0)^3 \right| = \max_{c \in [0,1]} \left| \frac{e^{-2c} \sin(c)}{6(5)} \right|.
  \]

- Since \( c \in [0,1] \), we know that \( 1 = e^0 \geq e^{-2c} \geq e^{-2} \), and \(-1 \leq \sin c \leq 1\). Hence
  \[
  \left| \frac{e^{-2c} \sin(c)}{6(5)} \right| = \frac{1}{30} \cdot |e^{-2c}| \cdot |\sin c| \\
  \leq \frac{1}{30} \cdot 1 \cdot 1 = \frac{1}{30}
  \]
  as required.

**Marking scheme:**

- 1 mark for writing that the error is bounded (in absolute value) by
  \[
  \max_{c \in [0,1]} \left| \frac{f'''(c)}{3!} \cdot (1 - 0)^3 \right| = \max_{c \in [0,1]} \left| \frac{c^3}{6(10 - c^2)} \right|.
  \]
  (or similar statement)

- 1 mark for explaining the bound and verifying that the error is still bounded above by \( 1/30 \). Be reasonably generous.

- The students lose 1 mark if they don’t have some explanation of the bound

**Long answer question — you must show your work**

3. [4 marks] Two particles move in the Cartesian plane. Particle \( A \) starts at \((2,0)\) and moves on the \(x\)-axis away from the origin at 1 unit per second. Particle \( B \) starts at the origin, and moves along the \(y\)-axis at 2 units per second (in the \(+y\)-direction). How fast is the distance between the particles increasing when \( A \) reaches \((6,0)\)?

**Solution:**

- Let \( A \)'s position by \((a,0)\) and \( B \)'s position be \((0,b)\). Then we know \( \frac{da}{dt} = 1 \) unit per second, and \( \frac{db}{dt} = 2 \) units per second. Note these rates are both positive.
• Our units are measured in seconds. Let the start time be \( t = 0 \). If \( z \) is the distance between the particles, we want to know \( \frac{dz}{dt} \) when \( a = 6 \).

• So, we need an equation relating \( a, b, \) and \( z \). Of course this equation is

\[
z^2 = a^2 + b^2
\]

and we differentiate with respect to \( t \):

\[
2zz' = 2aa' + 2bb'
\]

• To solve for \( \frac{dz}{dt} \), we need \( a \) and \( z \), when \( a = 6 \). It will take 4 seconds for \( A \) to reach this point, and in that time \( B \) moves 8 units. Hence \( a = 6, b = 8 \) and so

\[
z^2 = 6^2 + 8^2 = 100
\]

thus \( z = 10 \).

• Now we solve for \( \frac{dz}{dt} \):

\[
2zz' = 2aa' + 2bb'
20z' = 12 \cdot 1 + 2 \cdot 8 \cdot 2 = 12 + 32 = 44
z = \frac{11}{5} \text{ units per second}
\]

• Equivalently, we can write \( a = 2 + t \) and \( b = 2t \), so \( z^2 = a^2 + b^2 = (2 + t)^2 + (2t)^2 \). Then

\[
2z \frac{dz}{dt} = 2(2 + t) + 8t
\]

So

\[
2(10) \frac{dz}{dt} \bigg|_{t=4} = 2(2 + 4) + 8(4) = 44
\]

hence \( \frac{dz}{dt} = \frac{11}{5} \) units per second.

Marking scheme:

• 1 mark for obtaining the equation \( 2z(t) \cdot z'(t) = 2a \cdot a' + 2b \cdot b' \).

• 1 mark for \( a' = 1, b' = 2, b = 8 \) all correct.

• 1 mark for computing \( z(4) = 10 \).

• 1 mark for obtaining the correct answer \( z'(4) = \frac{11}{5} \).
**Very short answer questions**

1. \( \boxed{2 \text{ marks}} \) Each part is worth 1 marks. Please write your answers in the boxes. **Marking scheme**: 1 for each correct, 0 otherwise

   Consider a function, \( h(x) \), whose third-degree Maclaurin polynomial is \( 1 + 4x - \frac{1}{3}x^2 + \frac{3}{4}x^3 \).

   (a) What is \( h^{(3)}(0) \)?

   **Answer**: \( \frac{9}{2} \)

   **Solution**: The third Maclaurin polynomial for \( h(x) \) is

   \[
   h(0) + h'(0)x + \frac{h''(0)}{2} \cdot x^2 + \frac{h^{(3)}(0)}{6} \cdot x^3 = 1 + 4x - \frac{1}{3}x^2 + \frac{3}{4}x^3
   \]

   Thus \( h^{(3)}(0) = 6 \cdot \frac{2}{3} = 4 \) and for the next part, we note that \( h(0) = 0 \).

   (b) What is \( h''(0) \)?

   **Answer**: \( -\frac{2}{3} \)

**Short answer questions — you must show your work**

2. \( \boxed{4 \text{ marks}} \) Each part is worth 2 marks.

   (a) Estimate \( \sqrt[3]{26} \) using a linear approximation.

   **Solution**: We use the function \( f(x) = x^{1/3} \) and point \( a = 27 \) as the centre of our approximation since we know that

   \[
   f(27) = 3
   \]

   We compute \( f'(x) = \frac{1}{3}x^{-2/3} \); so

   \[
   f'(27) = \frac{1}{3} \cdot (27)^{-2/3} = \frac{1}{27}
   \]

   So, the linear approximation of \( 26^{1/3} = f(26) \) is

   \[
   T_1(26) = f(27) + f'(27) \cdot (26 - 27) = 3 - \frac{1}{27} = \frac{80}{27}
   \]

   **Marking scheme**:

   - 1 mark for writing that the function to use for the linear approximation is \( f(x) = x^{1/3} \), that \( a = 27 \). Also accept nearby \( a \) values that make it easy to compute \( f(a), f'(a) \).

   - 1 mark for obtaining the linear approximation \( 80/27 \). (or equivalent if they use other acceptable \( a \))
(b) Consider a function \( f(x) \) which has \( f^{(3)}(x) = \frac{e^{-x}}{8 + x^2} \). Show that when we approximate \( f(1) \) using its second degree Maclaurin polynomial, the absolute value of the error is less than 1/40.

**Solution:**

- The error is bounded (in absolute value) by
  \[
  \max_{c \in [-1,0]} \left| \frac{f'''(c)}{3!} \cdot (1 - 0)^3 \right| = \max_{c \in [-1,0]} \left| \frac{e^{-c}}{8 + c^2} \right| .
  \]

- When \( 0 \leq c \leq 1 \), we know that \( 1 \geq e^{-c} \geq e^{-1} \) and \( 8 \leq 8 + c^2 \leq 9 \), so
  \[
  \left| \frac{e^{-c}}{6(8 + c^2)} \right| = \frac{|e^{-c}|}{6|8 + c^2|} \leq \frac{1}{6|8 + c^2|} \leq \frac{1}{6 \times 8} = \frac{1}{48} < \frac{1}{40}
  \]
  as required.

- Alternatively, notice that \( e^{-c} \) is a decreasing function of \( c \), while for \( 0 < c \) \( 8 + c^2 \) is an increasing function of \( c \). Hence the fraction is a decreasing function of \( c \) and takes its largest value at \( c = 0 \). Hence
  \[
  \left| \frac{e^c}{6(8 + c^2)} \right| \leq \frac{1}{6 \times 8} = \frac{1}{48} < \frac{1}{40}.
  \]

**Marking scheme:**

- 1 mark for writing that the error is bounded (in absolute value) by
  \[
  \max_{c \in [-1,0]} \left| \frac{f'''(c)}{3!} \cdot (0 - (-1))^3 \right| = \max_{c \in [-1,0]} \left| \frac{e^{-c}}{6(8 + c^2)} \right| .
  \]
  (or similar statement)

- 1 mark for explaining why \( c = 0 \) is the right choice and then verifying that in that case the error is still bounded above by 1/48. Be reasonably generous.

- The students lose 1 mark if they don’t explain why \( c = 0 \) is the right choice (and they simply plug in \( c = 0 \), or compare the values they get between plugging in \( c = 0 \) and \( c = -1 \)).

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**Long answer question — you must show your work**

Two particles move in the cartesian plane. Particle A travels on the \( x \)-axis starting at \((10, 0)\) and moving towards the origin with a speed of 2 units per second. Particle B travels on the
y-axis starting at (0, 2) and moving away from the origin with a speed of 4 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point (5, 0)?

Solution:

• We compute the distance \( z(t) \) between the particle and the point with coordinates (0, 5) at each moment in time as

\[
z^2(t) = x(t)^2 + y(t)^2,
\]

where \( x(t) \) is the position on the x-axis of the particle A at time \( t \) (measured in seconds) and \( y(t) \) is the position on the y-axis of the particle B at the same time \( t \).

• We differentiate the above equation with respect to \( t \) and get

\[
2z \cdot z' = 2x \cdot x' + 2y \cdot y',
\]

or equivalently

\[
z \cdot z' = x \cdot x' + y \cdot y'.
\]

• We are told that \( x' = -2 \) and \( y' = 4 \). Further it will take 2.5 seconds for particle A to reach \( x = 4 \), and in this time particle B will reach \( y = 12 \).

• Alternatively (equivalently??) write \( x = 10 - 2t, y = 2 + 4t \) to get \( t = 5/2, x = 5, x' = -2, y' = 4, y = 12 \).

• At this point \( z = \sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13 \).

• Hence

\[
13z' = 5 \cdot (-2) + 13 \cdot (4) = -10 + 52 = 42
\]

\[
z' = \frac{42}{13} \text{ units per second.}
\]

Marking scheme:

• 1 mark for obtaining the equation \( 2z(t) \cdot z'(t) = 2x \cdot x' + 2y \cdot y' \).

• 1 mark for \( x' = -2, y' = 4, y = 12 \) all correct.

• 1 mark for computing \( z(2.5) = 13 \).

• 1 mark for obtaining the correct answer \( z'(5/2) = 42/13 \).