1. COMMENTS ON PROBLEMS IN HW#1

1-6: Everyone understood that $F_{s/t}$ was not smooth at the origin whenever $s \neq t$. The steps required to do the problem is to then take some atlas $A = \{(\phi_a, U_a)\}$ where we may assume that $\phi_a(U_a)$ is a unit ball at the origin, then create another Atlas $A_s$ (for any real $s > 0$) by:

a) Replace a SINGLE map in $A$, call it $\phi_0$, with the map $\tilde{\phi}_s = F_s(\phi_0)$ and leave its domain $U_0$ unchanged.

b) For $a \neq 0$, leave map $\phi_a$ the same but replace domains $U_a$ with $U_a \setminus \{\phi_0^{-1}(0)\}$.

Then conclude that:

c) the new system $A_s$ obtained above will be another Atlas: the domains still provide open cover of $M$, and the charts overlap smoothly. Indeed, the only point where they could not overlap smoothly would be at $\{\phi_0^{-1}(0)\}$ due to $F_s$ not being smooth at the origin, but $\{\phi_0^{-1}(0)\}$ is not in any overlap of domains by step (b).

d) The system $A_s$ is not compatible with $A_t$ for $s \neq t$: for the map $\tilde{\phi}_t \circ \tilde{\phi}_s^{-1} = F_t/s$ which is not smooth when $s \neq t$.

1-7 (d): This was mostly understood by everyone who did the problem, and amounted to writing the transitions in local coordinates $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ for overlapping coordinate domains. Namely, writing the transition as functions $y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n)$. A small, yet subtle mistake made by some was

a) Using either $n+1$ variables $x_1, \ldots, x_{n+1}$ or $y_1, \ldots, y_{n+1}$ in the above expression of the transition. Technically, these expression refer to the ambient space coordinates, and not coordinates on the projective spaces.

2.3 (c): Showing the map $F$ was smooth amounted, again, to writing it explicitly in local coordinates $x_1, x_2, x_3$ say on $S^3$ and coordinates $y_1, y_2$ say on $S^2$ (each coming for example from stereographic projection). Namely, writing $F$ in these coordinates as $y_1(x_1, x_2, x_3), y_2(x_1, x_2, x_3)$. Note that these are real valued functions, with real variables (there should not be any complex numbers in these functions). Most wrote these local functions correctly. The next step in the argument involves:

a) Observing that the functions $y_1(x_1, x_2, x_3), y_2(x_1, x_2, x_3)$ are smooth except at some points $(a, b, c)$, but arguing that such points are actually not in the domains of these functions (for example, if $x_1, x_2, x_3$ came from the stereographic projection map $\phi_N$ on $S^3$ “at the North pole”, then the points $(a, b, c)$ correspond to the North pole which is not even in the domain of $\phi_N$. An equivalent way to argue is to say that representing $F$ in a different choice of coordinates, say by replacing $\phi_N$ on $S^3$ with “$\phi_S$”, produces functions which are in fact smooth at the “bad points” above.

these points $(a, b, c)$

2.6 Similar to 1-7(d), this amounted to writing a coordinate representation of the map: explicitly, using a local coordinate $x_1, \ldots, x_n$ for the domain and a local coordinate $y_1, \ldots, y_k$ for the range then writing the corresponding coordinate expression of the map as $y_1(x_1, \ldots, x_n), \ldots, y_k(x_1, \ldots, x_n)$. A similar mistake can be made here too, as that in 1-7(d).
2.7 This was done well by most, by the same method as I suggested in class of producing bump functions \( f_1, ..., f_k \) (for arbitrary \( k \)) supported within disjoint coordinate balls \( B_1, ..., B_k \) centered say at \( p_1, ..., p_k \). One then argues their independence by observing:

a) If the function \( F = \sum_{i=1}^{k} a_i f_i(x) \) is the zero FUNCTION on \( M \), then for any \( j \) we have \( 0 = F(p_j) = a_j f_j(p_j) = a_j \). Thus the set \( f_1, ..., f_k \) is linearly independent as smooth functions on \( M \). But \( k \) was arbitrary, so we conclude the dimension of smooth functions on \( M \) is infinite.

b) An interesting alternative is to simply let \( f_j = (f^j) \) for some fixed non-constant bump function \( f \) on \( M \), then say the set \( \{ f_1, f_2, .... \} \) is independent “by the infinite dimensionality of the polynomials \( p(x) \)”. This is indeed a nice argument, but saying this and basically nothing more does not get full credit. Indeed, the independence of a set of real functions alone does not imply the independence of their compositions with \( f \). More precisely, it is the fundamental theorem of algebra (that any polynomial as at most as many roots as its degree), that gives the independence of their compositions. For any dependency in the \( f_j \)’s implies \( p(f) \) is the zero function on \( M \) for some non-trivial real polynomial \( p(x) \), which in turn implies \( p \) has infinitely many real roots since the range of \( f \) must contain the whole interval \([0, 1]\), which contradicts the fundamental theorem of algebra.