Lecture 8

(a) (wave equation):
\[ f_{tt} = c^2 f_{xx} \]

(b) (for some given const. \( a \),)
\[ f(x,t) = h(x+at) \]

Show that if \( h(x) \) is a free var. (bil) fun.

Then:
\[ f(x,t) = h(x+at) \]

Solves (1):
\[ f_{tt} = c^2 f_{xx} \]

\[ f_{t} = h'(x+at), \quad a \cdot c \]
\[ f_{xx} = h''(x+at), \quad a \cdot c \]
\[ f_{tt} = c^2 h''(x+at), \quad a \cdot c \]

\[ f_{t} = \frac{\partial f}{\partial x} \]
\[ f_{xx} = \frac{\partial^2 f}{\partial x^2} \]

\[ f_{tt} = \frac{\partial^2 f}{\partial x^2} \]

\[ f_{tt} = a^2 f_{xx} \]

\[ f_t = \frac{\partial f}{\partial t} \]
\[ f_{xx} = \frac{\partial^2 f}{\partial x^2} \]

\[ f_{tt} = \frac{\partial^2 f}{\partial t^2} \]

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\[ f_{tt} = \frac{\partial^2 f}{\partial t^2} \]

\[ f_{tt} = a^2 f_{xx} \]
Q: how are the curves related?

\[
\begin{align*}
&y = f(x, 0) \\
&y = f(x, 1) \quad \text{in } xy \text{ plane} \\
&y = f(x, 2) \\
&y = f(x, 3) \\
&\vdots
\end{align*}
\]

A: these are all translates of one another, moving in x dir, "like a wave."

* Can also write PDE's for 3-var functions \( f(x, y, z) \) as well.

for \( f(x, y, z) \) \( f_{xx} + f_{yy} + f_{zz} = 0 \) (Laplace)

for \( f(x, y, t) \) \( f_{tt} = a^2 (f_{xx} + f_{yy}) \) (wave equation)

* in general, hard to solve PDE's, just understand what they are and what it means to solve them.
**Differentiability**

Given, say, \( f(x,y) \):

**Q:** If \( f_x(a,b) \), \( f_y(a,b) \) both exist, then should we say \( f(x,y) \) is diff. at \((a,b)\)?

(similar quest. for \( f_x(y,z) \))

**A:** Yes is good guess, but consider:

\[
f(x,y) = \begin{cases} 
0 & \text{if } x = 0 \text{ or } y = 0 \\
1 & \text{otherwise}
\end{cases}
\]
We say \( f \) is differentiable at \((a, b)\) if:

\[
\begin{align*}
given \small \text{ small changes } dx, dy \text{ in } \\
\text{var. from } (a,b), \text{ Eq (1) gives } \\
"\text{very good" approx for conseq. } \\
\text{change in } f.
\end{align*}
\]

(VERY SIMILAR Stmt for \( f(x, y, z) \), \( df = f_x dx + f_y dy + f_z dz \))

\[ \text{-- etc} \]

What does "very good" mean here, ? Don't Dwell
on this; may see \{APEx
\}
\{CLP \} (Appendix)
defn. Just understand the idea behind (1),
and at least see how prev. ex. fails !
note that \( f_x(0,0), f_y(0,0) \) both exist! and are zero! But clearly, shouldn't call \( f(x,y) \) differentiable at \((0,0)\). (It is not even continuous at \((0,0)\)).

The "differential" of a function \( f \):

Let \( f(x,y) \) be defined around \((a,b)\) (and at \((a,b)\)). Suppose \( f_x(a,b), f_y(a,b) \) exist. Differential \( df(f(x,y)) \) at \((a,b)\) is the symbolic eqn:

\[
(1) \quad df = f_x(a,b) \, dx + f_y(a,b) \, dy
\]

(for 3 var, just add a term \( f_z(a,b,c) \, dz \) at end of (1)).
Relate differentials to linear approx:

\[
\text{df} = f_x(p) \, dx + f_y(p) \, dy + \ldots
\]

(differential \( \text{df} \) of \( f \) at \( p \), \( p = (a, b, \ldots) \))

\[
\begin{align*}
\Rightarrow & \quad f(x, y, \ldots) - f(a, b, \ldots) \\
\Rightarrow & \quad f(x, y, \ldots) \approx f(p) + f_x(p)(x-a) + f_y(p)(y-b) + \ldots
\end{align*}
\]

(linear approx of \( f(x, y, \ldots) \) at \( p = (a, b, \ldots) \))

\[
Z = f(p) + f_x(p)(x-a) + f_y(p)(y-b)
\]

(Tangent plane to graph of \( f \) at \( p = (a, b, f(a, b)) \))

\[
\text{ONLY FOR 2VAR, } x, y
\]
Linear approx of \( f(x,y,\ldots) \) is a linear function:
\[ Ax + By + C \] (or \( Ax + By + Cz + D \)), often written as
\[ L(x,y,\ldots) \], and here, \( f(x,y,\ldots) \) differentiable at \( p \) means:
\[ L(x,y,\ldots) \] is very close to \( f(x,y,\ldots) \) when \( (x,y,\ldots) \) close to \( p = (a,b,\ldots) \).

Target plane is just the graph of the linear approx!
\( f(x,y) \) diff. at \( p = (a,b) \) here means:
Tang. Plane behaves very much like graph of \( f(x,y) \) over pt. \( p = (a,b) \).

![Target plane to graph at pt.](image)
can be more precise here, by "behaves very much like," we mean: If we zoom into graph, at the pt (a, b, f(a, b)), then picture we see looks more & more like a plane (target plane) as we zoom in.

ex) Use differentials to estimate change in Vol. of cylinder.

\[ V(r, h) = \pi r^2h \]

Write differential, at (2.5):

\[ dV = V_r dr + V_h dh \]

\[ = (2\pi rh) dr + (\pi r^2) dh \]

\[ dV = 20\pi dr + 4\pi dh \]  \text{(at (2.5))}
So change in Vol is approx:

\[ 20\pi (.01) + 4\pi (.01) = .24\pi \]

* What if we just "took difference in V's"?

\(\text{ii})\) Change in Vol = \(V(2+dr, 5+dh) - V(2, 5)\)

= \(\pi (2+dr)^2(5+dh) - \pi 2^2 5 \)

= \(\pi (4 + 4dr + dr^2)(5+dh) - \pi 4 5 \)

= \( (20\pi dr + 4\pi dh) + (5\pi dr^2 + 4\pi drdh + \pi dh^2) \)

\[ "dV" \]

much smaller than \(dV\) when \(dr, dh\) small!

So yes, "\(dV\)" here is good approx of actual change in \(V\) when \(dr, dh\) are small.