Lect 21

proof of center of mass fmla ( X formula:

\[ \bar{x} = \frac{\int_{R} \int_{R} p(x,y) x \, dA}{\iint_{R} p(x,y) \, dA} \]  

\[ \bar{y} = \frac{\int_{R} \int_{R} p(x,y) y \, dA}{\iint_{R} p(x,y) \, dA} \]  

1. n point masses \( m_1, m_2, \ldots \) at \( x_1, x_2, \ldots \) balances at:

\[ \overline{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \]

Accept this.

2. mass distr on line "given by f(x)"

\[ dm = f(x) \, dx \]

balances at:

\[ \overline{x} = \frac{\int_{0}^{x} x \, dm}{\int_{0}^{x} dm} = \frac{\int_{0}^{x} x \cdot f(x) \, dx}{\int_{0}^{x} f(x) \, dx} \]

3. Lamina \( R \) in plane, "mass distributed by \( \rho(x,y) \)"

balances at:

\[ \overline{x} = \frac{\iint_{R} x \rho(x,y) \, dA}{\iint_{R} \rho(x,y) \, dA} \]
Since:

\[ dm = \int \rho(x,y) dy \, dx \]

"Collapse mass down to x-axis"
then apply (2); replace for with \( \int \rho(x,y) dy \)

TRIPLE INTEGRALS

Calculate these just as for double int.: successive antidiff:

\[ \int_{x}^{5} \int_{y}^{2} \int_{z}^{1} xy \, dx \, dy \, dz \]

ex)

\[ \int_{0}^{1} \int_{0}^{2} \int_{0}^{3} xy \, dx \, dy \, dz \]
\[ \begin{align*}
&= \int_{1}^{5} \int_{x}^{y} y \, dy \, dx \\
&= \int_{1}^{5} \left[ y^2 \right]_{x}^{y} \, dx \\
&= \int_{1}^{5} (y^2 - x^2) \, dx \\
&= \left[ \frac{y^3}{3} - \frac{x^3}{3} \right]_{1}^{5} \\
&= \frac{125}{3} - \frac{1}{3} \\
&= \frac{124}{3}
\end{align*} \]
will get a number as a result (a double int in y, z).

**Def**^n_{\text{Triple Integrals}}

A triple integral is expression of the form:

\[ \int_a^b \int_{h_2(y)}^{h_1(y)} \int_{g_2(y,z)}^{g_1(y,z)} f(x,y,z) \, dx \, dy \, dz \]

and is calculated, as before, by iterated integration (Result is a number.)

\* limits above indicate maximal allowable freedom to result in a number after calculating.

\* may of course, consider "different orders of integration" dx dy dz, dy dx dz, ... etc.
there are six possible orders !, must then
adjust variable dependencies in limits accordingly.

\[ \iiint_a^{b_2} f \, dy \, dx \, dz \]

**Meaning of Triple Integral**

As before, limits of \( z \) now represent a region, but now
in space; so we have \( \iiint_a^{b_2} g_2(x, u) h_2(u) \, dx \, du \)

\[ \iiint_a^{b_1} f(x, u, z) \, dz \, dx \, du \]

limits describe region \( R \) below:
\[
\begin{align*}
g_1(x,y) &\leq z \leq g_2(x,y) \\
h_1(y) &\leq x \leq h_2(y) \\
a &\leq y \leq b
\end{align*}
\]

So \( R \) is region \{ below \( z = g_2(x,y) \) sitting directly above \( z = g_1(x,y) \) \} region \( D \) in \( xy \) plane.

Think of \( D \) as projection of \( R \) onto \( xy \) plane.
* Can no longer think of int as "volumes under graphs". However, other interpretations still valid: may think of
\[ \iiint_{R} f \] as say, mass of region \( R \) with density function \( f(x,y,z) \).

This suggests

\[ \int_{a}^{b} \int_{c}^{d} f(x,y,z) \, dx \, dy \, dz = \int_{c}^{d} \int_{a}^{b} f(x,y,z) \, dx \, dy \, dz = \ldots. \]

Provided limits of int all represent same region \( R \) in space.
Can also approximate triple integrals with Riemann sums as helme.

1. Partition $R$ into $n^3$ subboxes.
2. Choose sample point $x_{ijk}$ in each box, consider "mass" of this subbox.
   \[ m = \frac{f(x_{ijk}) \cdot \Delta x \Delta y \Delta z}{n^3} \]
3. Add submasses:
   \[ \sum \sum \sum_{k=1}^{n^3} \sum_{j=1}^{n^3} \sum_{i=1}^{n^3} f(x_{ijk}) \Delta x \Delta y \Delta z \]
Theorem (Fubini)

If \( f \) is continuous on \( R \), Riemann sums approach limit as \( n \to \infty \).

Limit equals result of calculating \( \iiint f \, dv \) by iterated integration.

(appplies to more general \( f \) and \( R \) also)

Properties of Triple Integrals:

Same as for double integrals.

1. \( \iiint (f + g) \, dv = \iiint f \, dv + \iiint g \, dv \)

   etc.
\[ \iint_{R} \int_{0}^{z(x,y)} f(x,y,z) \, dz \, dy \, dx \]

\[ \iiint_{V} f(x,y,z) \, dx \, dy \, dz \]

\[ \iiint_{V} f(x,y,z) \, dV \]

\[ \iiint_{V} f(x,y,z) \, dx 
\]

\[ \iiint_{V} f(x,y,z) \, dy 
\]

\[ \iiint_{V} f(x,y,z) \, dz \]

\[ \iiint_{V} f(x,y,z) \, dV 
\]

\[ \iiint_{V} f(x,y) \, dx \]

\[ \iiint_{V} f(x,y) \, dy 
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\[ \iiint_{V} f(x,y) \, dz \]

\[ \iiint_{V} f(x) \, dx 
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\[ \iiint_{V} f(x) \, dy 
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\[ \iiint_{V} f(x) \, dz \]

\[ \iiint_{V} f(y) \, dx 
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\[ \iiint_{V} f(y) \, dy 
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\[ \iiint_{V} f(y) \, dz \]

\[ \iiint_{V} f(z) \, dx 
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\[ \iiint_{V} f(z) \, dy 
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\[ \iiint_{V} f(z) \, dz \]

\[ \iiint_{V} f(x^2 + y^2) \, dx \, dy 
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\begin{align*}
\text{a)} \quad & \int_{0}^{2} \int_{0}^{x} f \, dy \, dx + \int_{2}^{6} \int_{0}^{\sqrt{6-x}} f \, dy \, dx = \int_{0}^{2} \int_{0}^{6-y^2} f \, dx \, dy
\end{align*}
\]
\[ \int \int_{0}^{\sqrt{4-y^2}} \ln \left(1 + x^2 + y^2 \right) \, dx \, dy \]

Note:

\[ x^2 + 3y^2 = 4 - y^2 \]
\[ \Rightarrow 4y^2 = 4 \]
\[ \Rightarrow y = \pm 1 \]

So line intersects circle at height \( y = 1 \)

\[ = \int_{0}^{\pi/2} \int_{0}^{2} \ln (1 + r^2) \, r \, dr \, d\theta \]

\[ = \text{make subst. } u = 1 + r^2 \ldots \]