Lect 13

(a) Find eqn of tangent plane to surface:
\[ 2x^2 + y^2z = 3 \]
at \((1,1,1)\).

Method 1: Take \( z = F(x,y) \) implicitly defined.
Find \( F_x(1,1) \), \( F_y(1,1) \) by implicit diff.
Then use earlier eqn of tangent plane to graph of \( F \) at \((1,1,1)\).

Much quicker Method 2: Surface is \( f(x,y,z) = 2x^2 + y^2z = 3 \)
(level surface).
\[ \nabla f \cdot \mathbf{n} = 0 \]
\[ \text{Target plane has eqn } -2x + y - z = 0 \]
\[ f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1) = 0 \]

\[ \Rightarrow (2x^3) \big|_{(1,1,1)} (x-1) + (2y^2) \big|_{(1,1,1)} (y-1) + (6xz^2 + y^2) \big|_{(1,1,1)} (z-1) = 0 \]

\[ \Rightarrow 2(x-1) + 2(y-1) + 7(z-1) = 0 \]
Sketch $Df$ at $P, Q, R$ on diagram.

We used all properties of $Df$ here:

- $Df$ is perpendicular to level curve at $P$.
- $Df(P)$ points in dir of maximal rate of incr. of $f$ at $P$.
- $||Df(P)||$ is the maximal rate of incr. of $f$ at $P$.

(That's why $Df(R)$ is shortest...)
So why does $\nabla f(p)$ point perpendicular to the level surface ($3$ var) or curve ($2$ var) at $p$?

Illustration for $f(x, y)$ ($2$ var):

$\nabla f(x, y) = \mathbf{0}$; level curve of $f$

Start at $p$, move in dir $\mathbf{u}$.

$\Rightarrow$ "like moving along level curve"

$\Rightarrow$ $f$ does not change along path

$\Rightarrow$ $Df(p) = 0$

$\Rightarrow$ $\nabla f(p) \cdot \mathbf{u} = 0$

ie) $\nabla f(p)$ perp to $\mathbf{u}$
(c) Find Point (Plan C) on \( x^2 + y^2 = 1 \) (Surface)

[circle and plane diagram]

Wait to solve.
\[ \begin{align*}
0 &= a + t \\
0 &= b + 2bt \\
0 &= c + t \\
l &= a + b^2 + c
\end{align*} \]

Note:
\[ \nabla f(a, b, c) = \langle 1, 2b, 1 \rangle \]
- Target plane at \( P; \)
\[ (x-a) + 2b(y-b) + (z-c) = 0 \]

\( (0, 0, 0) \) is on line through \( P \)
with \( \nabla f(0) = \langle 1, 2b, 1 \rangle \)

\((a, b, c)\) lie on surface.

\[ \implies b = 0 \]
\[ a = c = -t \]
\[ \Rightarrow a = \frac{1}{2} \]
\[ \frac{b}{2} = \frac{1}{2} \]

\[ \begin{align*}
&b \neq 0 \\
t &= -\frac{1}{2} \\
a = \frac{1}{2}, \quad c = \frac{1}{2} \\
b &= 0
\end{align*} \]

\[ \left( \frac{1}{2}, 0, \frac{1}{2} \right) \]
Optimization (an application of all we learned)

Relative Maxima & Minima (of 2 var functions):

Def: Let $f(x,y)$ be defined around some $p = (a, b)$

1) $f$ has a relative max at $p$ if:
   
   $f(q) \leq f(p)$ for all $q$ in some disc around $p$

2) $f$ has a relative min at $p$ if:
   
   $f(q) \geq f(p)$
$f(x) = f(x, y)$

$f$ has

- relative max at $P$ ✓
- relative min at $Q$ ✓
- neither rel. max/min at $R$ ✓

($f$ has so-called "saddle" at $R$, formal definition coming soon)
If has rel. max at $P$ in all cases, note however,

<table>
<thead>
<tr>
<th></th>
<th>a)</th>
<th>b)</th>
<th>c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_x(P)$</td>
<td>0</td>
<td>DNE</td>
<td>0</td>
</tr>
<tr>
<td>$f_y(P)$</td>
<td>0</td>
<td>DNE</td>
<td>DNE</td>
</tr>
</tbody>
</table>
Conversely, if \( f_x(p) \) or \( f_y(p) \) exists and is non-zero, then \( f \) cannot have a rel max or min at \( p \).

We call \( p \) a critical point of \( f \) if

\[
\begin{cases} 
  f_x(p) = 0 \text{ or does not exist} \\
  f_y(p) = 0 \text{ or does not exist}
\end{cases}
\]

**Theorem (1st derivative Test)**

If \( f(x,y) \) has relative max or min at \( p \),

Then \( p \) is a critical pt. of \( f \).
So to locate rel max/min's, we first locate all critical pts. Then how do we tell if there is max or min at each?

**Theorem (2nd derivative Test)**

Assume $f(x,y)$ has critical pt. at $p = (a,b)$.

Assume $f_{xx}, f_{xy}, f_{yy}$ are continuous around $p$.

Let

$$D(x,y) = (f_{xx}f_{yy} - (f_{xy})^2)$$

1. $D(p) > 0$ ; $f_{xx}(p) > 0$ $\implies$ rel. min at $p$
2. $D(p) > 0$ ; $f_{xx}(p) < 0$ $\implies$ rel. max at $p$
3. $D(p) < 0$ ; then $f$ has "saddle" at $p$ ; neither max/min

Note: $D(p) = 0$ $\implies$ NO CONCLUSION! COULD STILL HAVE MAX/MIN at $p$. 
1. Find critical pts.

2. Classify critical pts.

3. Find max/min

f(x,y) = x^2 + xy - y^2

\n
\begin{align*}
0 &= 2x + y \\
0 &= x + 2y \\
x &= x - 2y
\end{align*} 

\begin{align*}
x &= 0 \\
y &= -x \\
x &= 0
\end{align*} 

(0,0)
2. Use 2nd Der Test!

\[ D(x,y) = f_{xx}f_{yy} - f_{xy}^2 \]

\[ = (2)(-2) - (1)^2 \]

\[ = -5 < 0 \]

So \( f \) has neither max nor min at \( P \), (has so called Saddle!)