Lect 12

* In prev. ex., does not make sense in (c), unless we restrict it to be unit vector.

\[ Df(p) = A \cdot \frac{\partial f}{\partial \mathbf{u}} \]

(only fair to restrict to unit vectors in (c))

We can summarize findings conveniently in terms of the gradient vector.

*
Let \( f(x,y) \) be differentiable at \((a,b) = p\).

1. Define gradient of \( f \) at \( p \) as the vector
   \[
   \nabla f(p) = \langle f_x(p), f_y(p) \rangle
   \]

2. Then Direct Deriv. of \( f \) at \( p \), in the \( \vec{u} \) dir. is:
   \[
   D_{\vec{u}}f(p) = \nabla f(p) \cdot \vec{u}
   \]
   (Just same formula as before)

3. Maximum Value
   \[ \frac{\partial}{\partial t} D_{\vec{u}}f(at, bt) \] occurs when
   \[ \vec{u} = \frac{\nabla f(p)}{||\nabla f(p)||} \]
   \[
   \]
   Minimum Value
   \[ \frac{\partial}{\partial t} D_{\vec{u}}f(at, bt) \] occurs when
   \[ \vec{u} = -\frac{\nabla f(p)}{||\nabla f(p)||} \]
   (Assume \( \nabla f \neq 0 \))
Given \( f(x, y, z) \), \( \frac{df}{d\mathbf{v}} \) at \( p = (a, b, c) \), definitions are identical, just add another component where needed:

1. **Gradient of** \( f(x, y, z) \) **at** \( p \):
   \[
   \nabla f(p) = \langle f_x(p), f_y(p), f_z(p) \rangle
   \]

2. **Pit. Der.**
   \[
   D_{\mathbf{U}} f(p) = \nabla f(p) \cdot \mathbf{U}
   \]

3. Same set as before

\( \sum \)
\( f(0,0) = 0 \) (ii) find \( f\left(x, (0,0)\right) \)

\( \frac{2\pi}{11} = m \) (i) find \( f\left(x, (0,0)\right) \), \( f\left(\alpha, (0,0)\right) \)

\( f\left(\alpha, (0,0)\right) \)

\( \frac{2\pi}{11} = m \) (ii) find \( f\left(x, (0,0)\right) \)

\( a = -\frac{\sqrt{3}}{3} \)
A function $T(x, y) = \begin{cases} 1 & x = 1 \\ 2 & x = 2 \\ 3 & \text{otherwise} \end{cases}$.

Find $\nabla T(p)$ at $p(2, 1, 1)$.

So, $\nabla T(p) = (0, 1, 2)$.

Bee starts at $P$, flies along with the vector towards $Q(3, 2, 2)$. Rate of change of temp? (Bee feels)

\[ \frac{\text{dx}}{\text{dt}} = T(2, 1, 1) \]
c) Let \( S(\mathbf{x}, y, z) = x + z \). Find \( \mathbf{D}_w S \) both ways.

Solution: want \( \mathbf{D}_w S = 0 \)
\[ D(3, 4) = 0 \]
\[ 7S(p) = 0 \]
\[ <a_0, b_0, c_0> = 0 \]
\[ <1, 2, 3> \cdot <a_1, b_1, c_1> = 0 \]
\[ <1, 0, 1> \cdot <a_2, b_2, c_2> = 0 \]
\[ a + 2b + 3c = 0 \]
\[ a + c = 0 \]
\[ \text{let } c = t \]
\[ a = -t \]
\[ b = -3t + t = -2t \]
\[ w = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \]
\[ \cos \theta = \frac{1}{\sqrt{3}} \]
\[ \theta = \frac{\pi}{3} \]
d) (extra) i) In what dir is $T$ changing most rapidly at P?

ii) What is rate of change in this direction?

\[ \nabla T(P) = \left< 1, 2, 3 \right> \]

\[ \frac{1}{\sqrt{14}} \]

ii) \[ \frac{\nabla T(P)}{||\nabla T(P)||} \cdot \left( \frac{\nabla T(P)}{||\nabla T(P)||} \right) = \frac{1}{||\nabla T(P)||} \]

\[ = \frac{1}{\sqrt{14}} \]
Geometry & gradient vectors:

\[ \nabla f \text{ tells us about geometry of level curves of } f \\
\text{(or level surfaces if } f \text{ has 3 vars)} \]  

1. Given \( f(x,y) \) diff. at \( p(a,b) \). Then \( \nabla f(p) \) is perpendicular to level curve \( y = f \) at \( p \).  
   (i.e. perp. to tangent line to level curve \( y = f \) at \( p \) )

2. Given \( f(x,y,z) \) diff. at \( p(a,b,c) \). Then \( \nabla f(p) \) is perpendicular to level surface \( y = f \) at \( p \).  
   (i.e. perp. to tangent plane to level surface \( y = f \) at \( p \) )
1. Show what is meant:

\[ \nabla f(p) \]

\[ p = (a, b) \]

\[ f(xy) = C \]

2. Target (sur)

\[ f(xy, z) = \Delta \]

\[ p = (a, b, c) \]

Target plane
In $\mathbb{R}^3$ we refer to "tangent plane to surface given by $f(x,y,z) = d$".

(but only defined tangent planes to graphs $z = F(x,y)$ !)

However, may always think $\partial$:

at $Q$ sum $\frac{\partial}{\partial z} z = F(x,y)$ defined implicitly by $f(x,y,z) = d$

at $R$ sum

or $\frac{\partial}{\partial y} y = F(x,z)$

or $\frac{\partial}{\partial x} x = F(y,z)$
Is this same question as before?

Yes, I found it to be a triangle with

Base vectors: \((a,b,c)\)

Since plane equations: \(p(a,b,c)\)

\[
\begin{align*}
0 &= \frac{x}{p} + \frac{y}{(y-b)} + \frac{z}{(z-c)} \\
\end{align*}
\]

At \(P(a,b,c)\) this plane meets:

At \(P(a,b,c)\) the surface given by

\(f(x,y,z) = \text{a function of } x, y, z\)

Tangent plane to surface given by

And so, in particular:
Given graph \( Z = F(x, y) \)

Target plane at \((a, b, F(a, b))\) ?

Use (1) by:

\[
Z = F(x, y)
\]

\[
\Rightarrow f(x, y, z) = z - F(x, y) = 0
\]

\[
\Rightarrow \text{T. Plane to } f(x, y, z) = 0 \text{ has eqn:}
\]

\[
f_x(p)(x-a) + f_y(p)(y-b) + f_z(p)(z-F(a, b)) = 0
\]

\[
\Rightarrow -F_x(a, b)(x-a) - F_y(a, b)(y-b) + (z-F(a, b)) = 0
\]

\[
\Rightarrow Z = F(a, b) + F_x(a, b)(x-a) + F_y(a, b)(y-b)
\]

Same formula as before!