Lect 6

Ex) find equation of plane containing
\[ (1, 1, 1), (1, 2, 1), (2, 1, 2) \]

A normal vector \( \mathbf{n} \) is:
\[
\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 1 & 1 \\
1 & 1 & -1 \\
\end{vmatrix} = \langle 1, 0, -1 \rangle
\]

A point on plane \( \mathbf{P} \) is \( (1, 1, 1) \)

Equation of plane:
\[
(x-1)-(2-1)=0
\]

\( x - z = 0 \)
Do lines \[ \begin{cases} x = 0 \\ y = 2 + s \\ z = 2s \end{cases} \quad \begin{cases} x = t \\ y = 1 - t \\ z = 1 + 3t \end{cases} \]

intersect?

Try to solve \( 0 = t \) \( \text{for} \ s,t \).

\[ \begin{align*}
2 + s &= 1 - t \\
2s &= 1 + 3t
\end{align*} \]

\[ 2 + s = 1 \quad \Rightarrow \quad s = -1 \]

\[ s = \sqrt{2} \]

No solution

Lines don't intersect.
9) Do planes \(3x + 2y + z = 1\) and \(x + y = 0\) intersect? At what angle?

Solve \(\begin{cases} 3x + 2y + z = 1 \\ x + y = 0 \end{cases}\)

\[\Rightarrow \begin{cases} x = t \\
y = -t \\
z = 1 - t \end{cases}\] (Just let \(x = t\))

The line has symmetric equations:

\[x = -y = 1 - z\]

Planes intersect in a line given by above parametric equations!
Since $\mathbf{n}_1 \cdot \mathbf{n}_2 > 0$, hence angle between these $\leq \pi/2$

*Notes: $\theta$ is same as angle between $\mathbf{n}_1, \mathbf{n}_2$

$\theta = \text{arc} \cos \left( \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{||\mathbf{n}_1|| \cdot ||\mathbf{n}_2||} \right) = \frac{\pi}{\sqrt{144 + 99}}$

Planes intersect in line:

$\mathbf{n}_1 = \langle 3, 3, 0 \rangle$
$\mathbf{n}_2 = \langle 1, 1, 0 \rangle$

(obtained by inspecting plane equations.)
Given 2 planes $\Pi_1, \Pi_2$ with normal $\vec{n}_1, \vec{n}_2$.

- Planes are parallel if $\vec{n}_1 \parallel \vec{n}_2$.
- If planes intersect, angle $\theta$ between them is

$$\theta = \arccos \left( \frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\| \|\vec{n}_2\|} \right)$$

Given 2 lines $L_1, L_2$ with dir $\vec{v}_1, \vec{v}_2$.

- Lines are parallel if $\vec{v}_1, \vec{v}_2$ parallel
- Lines are called skew if not parallel.
now, some std "distance formulae":

A) distance from a pt. \( P \) to a plane \( \overrightarrow{\Pi} \) is:
\[
d = \| \text{proj}_{\overrightarrow{\Pi}} \overrightarrow{PA} \|
\]
\( (Q \text{ is any pt. on plane, } \overrightarrow{\Pi} \text{ is normal vector}) \)

B) distance from a pt. \( P \) to a line \( \overrightarrow{v} \) is:
\[
d = \frac{\| \overrightarrow{v} \times \overrightarrow{PA} \|}{\| \overrightarrow{v} \|}
\]
\( (Q \text{ is any pt. on line, } \overrightarrow{v} \text{ is direction of line}) \)
c) distance from line \( L_1 \) to another line \( L_2 \)

\[ d = \| \text{proj}_{\vec{v}_2} \vec{PQ} \| \]

(assume \( \vec{v}_1 \) not parallel to \( \vec{v}_2 \))

\( \vec{P} \) any pt. on \( L_1 \), \( i \) \( \vec{v}_1 \) dir. for \( L_1 \)

\( \vec{Q} \) any pt. on \( L_2 \), \( i \) \( \vec{v}_2 \) dir. for \( L_2 \)

Best not to bother memorizing these, but understand why they are true from the pictures!

Will not even really use these in later sections!
--- hardest to see is C). For this, imagine 2 parallel planes \( \Pi_1, \Pi_2 \), where \( \Pi_1 \) contains \( L_1 \), \( \Pi_2 \) contains \( L_2 \). Then normal for these is

\[
\vec{N} = \vec{V}_1 \times \vec{V}_2
\]

(since \( \vec{V}_1, \vec{V}_2 \) are in both \( \Pi_1, \Pi_2 \))

So picture then becomes:
distance between $L_1, L_2$ is just dist. between $\Pi_1, \Pi_2$, which is $\|\text{proj}_{\hat{n}} PQ\|$
* there are, in fact, several different ways to determine distances in A, B, C...

eg) distance between L₁, L₂ from earlier example? Can use formula in C).

Another approach:
Find Point Q on L2 so that

$Q = \left( t, \frac{-t+3-1}{3}, \frac{3t-25+1}{5} \right)$

$\cdot \{t, \{1,0,0\} \}$

$\cdot \{1, -1, 3\}$

$= 0$
\[
\begin{align*}
&-2(t+s+1) = 0 \\
&-2(t+s+1) + 3(3t-2s+1) = 0
\end{align*}
\]

\[
\begin{align*}
-2t-2s &= 2 \\
11t - 5s &= -4
\end{align*}
\]

\[
\begin{align*}
-11t - 22s &= 22 \\
11t - 5s &= -4
\end{align*}
\]

\[
-27s = 18 \\
\therefore s = \frac{18}{-27}
\]

\[
t = \left(\frac{-4 + 5\cdot\frac{18}{-27}}{-27}\right)
\]

So, not nice numbers... but still, we found s, t! Now put these back into our original equation to get P, Q... then answer is...
Lect 7

Quadratic Surfaces (10.1)

Sketching by "traces"

Next, both discuss Quadric Surfaces

Both actually starts by discussing cylinders or more variables missing

We already learned how to sketch those
these are the analogues of conic sections in plane (parabola, hyperbola, ellipses).

Both focuses on those having equations:

\[
\begin{cases}
    z^n = Ax^2 \pm By^2 \\
    Ax^2 + By^2 \pm z^2 = \pm 1
\end{cases}
\]

\(n=1,2\)
\(A,B>0\)

Key is not to memorize these, or any other 'forms', but to know technique for sketching these!
6) Use part (a) to sketch curve.

Sketch the trace in x-y coordinate.

With y = constant, please locate:
(a) how do intersections & surface

\[ z = y - \frac{x^2}{2} \]

(c) Sketch surface in P3 with constant
(a) \[ z = 4 - x^2 \] \[ z = 1 - x^2 \] \[ z = x^2 \] \[ z = -x^2 \]

Inters. of \( S \) with planes \( y = k \) \[ k = 0 \] \[ k = 1, -1 \] \[ k = 2, -2 \]

Call these the "\( y \) traces" of surface.
What is the profile of a saddle? (intersects with...
a) What do "2 traces" look like in 3D?

Interset, $S$ with plane $Z = k$

$k = \frac{\sqrt{2}}{4} - x^2$

$k < 0$

$k = \frac{\sqrt{2}}{4} - x^2$

$k > 0$
Could use $z$ traces to sketch $S$.

But this is harder to visualize than $x$ or $y$ traces.

How to know which one to use? $S$ gives ellipses. $x$ gives parabolas, $y$ gives hyperbolas. In order, these are:

- Ellipses
- Parabolas
- Hyperbolas

We can neglect curves.
$z = k$

traces

$x^2 + y^2 = k - 2$

$k \geq 2$
Lect 8

Functions of several variables

* Domain, Range of \( f(x, y) \), \( f(x, y, z) \)

* Graphs of \( f(x, y) \)

* Level curves of \( f(x, y) \)

* Level surfaces of \( f(x, y, z) \)

Course grade depends on:
- Homework average, test average
- Final exam average

Altitude of terrain at a point on a map depends on:
- \( x \) coord. of pt., \( y \) coord. of pt.
Altitude at a plot on map depends on longitude, latitude.

Body Mass Index depends on height, weight.

Course grade depends on Webwork, quizzes, test, exam.

Temperature at a plot in atmosphere depend on longitude, latitude, altitude.

All on and/or of function of several variables.
functions of 2 variables, $f(x,y)$

* Domain, Range

\[
\begin{align*}
D & : \text{Domain of } f(x,y) \text{ is all } (x,y) \in \mathbb{R}^2 \\
R & : \text{Range of } f(x,y) \text{ is all values attained by } f.
\end{align*}
\]

e.g.) Domain and Range of $f(x,y)$

1) $f(x,y) = x^2 + y^2$

iii) $f(x,y) = \frac{\sqrt{y-x^2}}{1-x^2}$
ii) Domain: $x, y \geq 0$, can sketch in plane.

Range: $[0, \infty)$.
(5)

iii) $y = x^2 - x - 1 \geq 0 \Leftrightarrow x^2 - x - 1 \geq 0$

Sketch/shade domain in plane.

Range: $(-\infty, -1 \cup [2, \infty)$
Graph of $f(x,y)$

This is the surface $S$ in $\mathbb{R}^3$ with equation

$z = f(x,y)$

(lies entirely over domain $\partial f$)
2a) Graph of \( f(x, y) = x^2 + y^2 + 2 \)

2b) Sketch surface \( z = x^2 + y^2 + 2 \)

*(Did this one!)*

Level curves of \( f(x, y) \) are just concentric circles in \( x, y \) plane!
level curves of \( f(x,y) \)

The "\( k \)-level curve" of \( f(x,y) \) is just the curve with equation

\[ k = f(x,y) \]

in the \( x,y \) plane.

(In other words, it is just the \( z \) trace, or the intersection of graph with the plane \( z = k \).)

\[ \text{eg) sketch the level curves of} \]

\[ \begin{array}{ll}
  (i) & f(x,y) = \sqrt{x+y^2} \\
  (ii) & f(x,y) = x^2 + y^2 \\
\end{array} \]
\[
\begin{align*}
\kappa &= \frac{1}{2} (3+\sqrt{3}) \\
\kappa &= \frac{1}{2} (3-\sqrt{3}) \\
X^2 + \frac{1}{3} &= \kappa \\
X &= \sqrt{\kappa - \frac{1}{3}}
\end{align*}
\]