Lect 2.5

In an earlier example, \( \text{gg} = \text{gg} \).

Looks like:

\[ \Gamma = 2 \cos \theta \]

\[ (x-1)^2 + y^2 = 1 \]

\[ r = 2 \cos \theta \]

\[ 0 = \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{3\pi}{2} \]

Note:
Applications

Mass & Centers of Mass

Let $D$ be a "lamina" in $xy$ plane: a "material sheet" in plane

Suppose density over $D$ is non-uniform, and given by a function

$$\rho(x, y) \geq 0 \text{ on } D$$
ie) then a little rect \( \int \int \rho \, dA \) at \((xy)\)

has mass

\[
dm = \rho(x,y) \, dA
\]

\[
= \rho(x,y) \, dx \, dy
\]

(so \( \rho(x,y) \) here is density at \((xy)\) and has units like \(g/cm^2\) in example)

\[\therefore \text{ Total Mass of } D\]

\[
m = \iiint_D dm = \iiint_D \rho(x,y) \, dA
\]

(just another interpretation of double integral)
Center of mass:

Let $D, p(x,y)$ be as before, define

\[
\begin{cases}
    M_y := \iint_D x \, p(x,y) \, dA \\
    M_x := \iint_D y \, p(x,y) \, dA
\end{cases}
\]

Then center of mass of $D$ is:

\[
(\bar{x}, \bar{y}) = \left( \frac{M_y}{m}, \frac{M_x}{m} \right)
\]
Lamina balances at \((x, y)\), as shown.

We just "know" it is at \((0, 0)\).
--- to actually find this with formulae:

\[ M_y = \iiint_D x \, d\mathbf{A} = \iiint_D x \, d\mathbf{A} = 0 \]

\[ M_x = \iiint_D y \, d\mathbf{A} = 0 \]

Could actually do this.

Int. by iterated Int.

OR note: \( f(x,y) = x \)

is antisymmetric in \( x \)

\((\text{add})\) \( f(x,y) = -f(-x,y) \)

and \( D \) is symmetric about \( y \)-axis

\[ \Rightarrow (\overline{x}, \overline{y}) = (0, 0) \]

as expected!
(d) Find center of mass \( \bar{y} \): \[ x^2 + y^2 = 1 \] uniform density \( \rho = 1 \)

Intuitively, as observed before, see that

\[ \bar{y} = 0 \]

But what about \( \bar{x} \)?
\[ M_x = \iint_D y \, dA \]
\[ = \int_0^1 \int_0^{\pi/3} r^2 \sin \theta \, dr \, d\theta \]
\[ = \int_0^{\pi/3} \left[ \frac{r^3}{3} \sin \theta \right]_0^1 \, d\theta \]
\[ = \frac{1}{3} \left( -\cos \theta \right) \bigg|_0^{\pi/3} \]
\[ = \frac{1}{3} (-\cos \frac{\pi}{3} + \cos 0) = \frac{2}{3} \]

\[ \bar{y} = \frac{M_x}{m} = \frac{2/3}{\pi/2} = \frac{4}{3\pi} \]
Proof of center of mass formula:

Mass distributed along a line.

Mass is \( dm = \rho(x) dx \) at \( x \).

Continuous density \( \rho(x) \) at \( x \).

\[ \text{balances at } x = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \]

\[ \text{balances at } x = \frac{\int x \rho(x) dx}{\int \rho(x) dx} = \frac{\int x \rho(x) dx}{m} \]
Continuous mass distribution in plane, density $\rho(x,y)$ at $(x,y)$.

Strip has mass $\int \rho(x,y) \, dy \, dx$.

This 'balances at' $\int x(\int \rho(x,y) \, dy) \, dx = \int \int x \rho(x,y) \, dy \, dx = \frac{\int x \rho \, dA}{m}$. 

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--- can think of as saying, material sheet (lamina) balances on the line \( x = \frac{\int x \rho dA}{m} \)

\[ y = \frac{\int y \rho dA}{m} \]

\[ z \]

--- likewise, will also balance on line \( x = \frac{\int x \rho dA}{m} \)

--- so it balances at pt. (center of mass) \( (\frac{\int x \rho dA}{m}, \frac{\int y \rho dA}{m}) \)!!
Lect 26

Surface Area

Given graph $z = f(x, y)$ over a region $R$.

$z = f(x, y)$
To find surface area \( S \) of this patch \( B \) of the graph, will do as follows:

1. Consider small patch at \((x, y) \in \mathbb{R} \) with dimensions \( dx \), \( dy \), \( dz \).

2. Think of small patch of surface directly above this small rect.
   It will be like parallelogram spanned by the two vectors
   
   \[
   \mathbf{A} = \langle dx, 0, 0 \rangle, \quad \mathbf{B} = \langle 0, dy, 0 \rangle.
   \]
The area of the small patch is:

\[ dS = \left| \begin{array}{cccc}
\vec{i} & \vec{j} & \vec{k} \\
\partial x & 0 & f_x dx \\
0 & \partial y & f_y dy
\end{array} \right| \\
= \left\| \langle -f_x dx dy, -f_y dx dy, dx dy \rangle \right\| \\
= \sqrt{f_x^2 + f_y^2 + 1} \ d\partial x \partial y \\
\]

And so!! Surface Area of Whole Patch is:

\[ \text{Area} = \iiint dS = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \ dA \]
Calculate the volume of the solid.

\[
\frac{h}{2} - x = -\sqrt{1 - x^2 - y^2}
\]

\[
f(x) = \frac{h}{2} - \sqrt{1 - x^2 - y^2}
\]

\[
f_y = \frac{-y}{\sqrt{1 - x^2 - y^2}}
\]

\[
\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{0}^{\sqrt{1 - x^2}} f_x \, dy \, dx + \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\sqrt{1 - x^2}}^{0} f_y \, dy \, dx
\]

The volume is 13.
--- DO THIS, use subst. \( \begin{align*}
u &= 1 - r^2 \\
\frac{du}{dr} &= -2r \\
\end{align*} \)

--- GET \( 2\pi \) as answer.
SOME PAST FINAL (Double Int.)

\[ \int_{0}^{1} \int_{0}^{x^2} x^3 \sin y^3 \, dy \, dx \]

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\[ \int_{0}^{\sqrt{2}} \int_{0}^{x^2} x^3 \sin y^3 \, dx \, dy = \frac{1}{4} \int_{0}^{1} y^2 \sin y^3 \, dy \]

a) Sketch!

b) Calculate!
Region in plane above $x = \sqrt{3}$, below $y = 1 - x^2$

a) Sketch!

b) Write $\iint_R f(x,y) \, dA$ in orders $dx \, dy$, $dy \, dx$

\[
\int_{-1}^{\sqrt{1-y}} \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y) \, dx \, dy + \int_{\sqrt{1-y}}^{1} \int_{0}^{\sqrt{1-x^2}} f(x,y) \, dy \, dx
\]

\[
\int_{0}^{\sqrt{1-y}} \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y) \, dx \, dy + \int_{\sqrt{1-y}}^{1} \int_{0}^{\sqrt{1-x^2}} f(x,y) \, dy \, dx
\]

c) Find Int. when $f(x,y) = e^{x - x^{3/3}}$
\[ \int_{-1}^{1} \int_{0}^{1-x^2} e^{x-x^{3/3}} \, dy \, dx \]

\[ = \int_{-1}^{1} (1-x^3) e^{x-x^{3/3}} \, dx \]

\[ = \int_{-2^{1/3}}^{2^{1/3}} e^u \, du \]

\[ = e^u \bigg|_{-2^{1/3}}^{2^{1/3}} \]

\[ = e^{2^{1/3}} - e^{-2^{1/3}} \]
Region bounded by $X=2 \quad \frac{1}{3} x^2 + y^2 = 16$ (smaller part)

$x=2$
$\Rightarrow r \cos \theta = 2$
$\Rightarrow r = \frac{2}{\cos \theta}$

\[
\iiint_R (x^2 + y^2)^{-3/2} = \iiint_{R} r^{-3} \cdot r \, dr \, d\theta
\]

\[
= \left[ -r^{-1} \right]_{r=2/\cos \theta}^{r=r_{\pi/3}}
\]

\[
= \int_{\pi/3}^{\pi/3} \left( \frac{1}{4} + \frac{\cos \theta}{2} \right) \, d\theta
\]

\[
= \frac{\pi}{4} + \frac{\sin \theta}{2} \bigg|_{\theta=\pi/3}^{\theta=\pi/3} = \text{YOU FINISH...}
\]