Lect 22

previous calculation similar to derivation of volumes of revolution in Calc I.

disc: $dV = \pi f(x)^2 \, dx$

**Volume** = $V = \int_a^b \pi f(x)^2 \, dx$    "standard" formula for Vol. of revolution.
* Changing order in iterated integrals...

Eq) "Change order" of following iterat. int.

a) \[ \int_0^1 \int_0^{x^2} x^2 y \, dy \, dx \]

b) \[ \int_0^1 \int_0^{e^{x^2}} x \, dx \, dy \]

c) \[ \int_0^2 \int_0^{\sqrt{y}} x \, dx \, dy \]
2. Changing order of integration in following.

\[ \int_0^1 \int_0^2 x^2 \, dy \, dx \]

(a) \[ \int_0^2 \int_0^1 x^2 \, dy \, dx \]

(b) \[ \int_0^1 \int_0^2 x^2 \, dx \, dy \]
a) \[
\int_{0}^{1} \int_{0}^{x^2} x^2y \, dx \, dy
\]

Here, limits of int are simply switched! Result will be same in either case.

b) \[
\int_{0}^{1} \int_{0}^{1} e^{x^2} \, dy \, dx
\]

↑ note, here \(\int \int\) is wrong!

It would not even yield a number!

But a fun of \(y\) instead!

Must look at region of int in \(xy\) plane.
$$y = x$$

$$\Rightarrow \int = \int_0^1 \int_0^{e^x} e^{x^2} \, dy \, dx$$

Note: Integrals should give same answer, but first integral cannot even be started. ...

\[ \int e^{x^2} \, dx = ? \]

However,

\[
\int \int_0^1 e^{x^2} \, dy \, dx = \int_0^1 ye^{x^2} \bigg|_{y=0}^{y=x} \, dx = \int_0^1 x e^{x^2} \, dx
\]

\[
= \frac{e^{x^2}}{2} \bigg|_0^1 = \frac{e}{2} - \frac{1}{2}
\]
c) \[
\int_0^1 \int_0^{x^2} x \, dy \, dx + \int_1^2 \int_0^1 x \, dy \, dx
\]

Note: 2 separate integrals required here!

Note: Could do these problems, in fact, without any sketches, but usually involves ad hoc manipulation of inequalities.
\[ \begin{align*}
\sqrt{y} & \leq x \leq 2 \\
0 & \leq y \leq 1
\end{align*} \Rightarrow \begin{cases} 
0 \leq y \leq \frac{1}{2} \quad \text{and} \quad x^2 \\
0 \leq x \leq 2
\end{cases} \]

Want to pick the smaller one, which is smaller? Depends on \( x \)!
13.2 Double integrals

A different approach to volumes under graphs $z = f(x, y)$.
1. partition $D$ into $n$ subrectangles, $\Omega$
   equal dimension $\Delta x \times \Delta y$ say.

2. take apt. $(x_i, y_i)$ inside $i$th subrect and
   raise a rect. box over this of height $f(x_i, y_i)$
Let $f(x,y)$ be continuous on region $D$ in plane.

We define the **Double integral** \( f \) over $D$ as:

\[
\iiint_{D} f(x, y) \, dx \, dy
\]

where $D$ refers to the region $\iiint_{D}$.

(1) Such sums are called **Riemann Sums**.

**Definition:**

For fixed $D$, 

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta x \Delta y
\]

is the **Double Integral** $\iiint_{D} f(x, y) \, dx \, dy$.
Box has volume $V_i = f(x_i, y_i) \Delta x \Delta y$

3. Add up all sub-volumes

$$\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} f(x_i, y_i) \Delta x \Delta y$$

* Above sum approximates volume under graph.

We can define volume to be limit of this process:

Taking finer and finer subdivision of $D$ into rects, raising thin boxes over each up to graph, then adding up volume of these!

In general, we define:
A: YES... In general, we have...

Q: Are these really the same?

Intersect intervals

In 2 different ways:

Deformed intervals -- limit process

Note: We seem to have derived different values

Also, limit here is independent of choice of path (i.e., v1) inside it's support.

This continues, eventually does approach a limit; provided we are assuming here that process

Note, we are assuming here, that process...
Theorem (Fubini thm)

Let $f(x,y)$ be continuous on $D$ in plane.

1. If $D : \{ g_1(y) \leq x \leq g_2(y) \quad \text{then} \quad a \leq y \leq b$

$$\int \int_D f(x,y) \, dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y) \, dx \, dy$$

2. If $D : \{ g_1(x) \leq y \leq g_2(x) \quad \text{then} \quad a \leq x \leq b$

$$\int \int_D f(x,y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx$$
\[ e^x \]

Using partition \( P \) into 4 subintervals, (Riemann sum with \( n = 4 \))

\[ R \]

approx

\[ \int_a^b f(x) \, dx \]

\[ \sum x_{i-1} \Delta x \]

Trapezoidal rule

\[ \int_a^b g(x) \, dx \]

be calculated as 

\[ \text{iterated int} \]

triple of this as saying double int

can be iterated int

\[ \text{approximate} \]

iterated int (5) interchangingly when

\[ \text{hence forth, may use terms double int} \]
\[
\frac{1}{4} (1 + 2a + 2b)
\]

\[
\begin{align*}
\left( \frac{3}{4} \right)_{a+b} &+ (\frac{3}{4})_{2a+2b} + \left( \frac{3}{4} \right)_{a+b} = 1
\end{align*}
\]

\[
\int_{x^2+y^2 \leq 4} \left( a^2 + b^2 \right) = 4
\]

\[
\sum_{i=1}^{n} x_i y_i = \text{Sum of products}
\]

\[
\text{Number of points in the unit square}
\]

\[
\text{Sum of squares}
\]

\[
\text{Number of points in the unit square}
\]
Express \( \iint_{R} f(x,y) \, dA \) as iterated integral(s):

\[ \int_{0}^{1} \int_{\frac{x^2}{4}}^{\sqrt{1-x^2}} f(x,y) \, dy \, dx \]

\[ = \int_{0}^{1} \int_{\frac{x^2}{4}}^{\frac{1}{4} - (y - \frac{1}{2})^2} f(x,y) \, dy \, dx + \int_{\frac{1}{4} - (x - \frac{1}{2})^2}^{\sqrt{1-x^2}} f(x,y) \, dx \, dy \]
express $\int \int f(x,y) \, dA$ as iterated integral(s) by partitioning $R$. Note: It looks like 8 iterated int. are needed.

As could say

$$\int \int f(x,y) \, dA = \int \int f(x,y) \, dA - \int \int f(x,y) \, dA$$

$R$ $R_1$ $R_2$ (additive property of double int.)
(Assuming \( f(x, y) \) is defined in \( \mathbb{R}^2 \). That is \( f(x, y) \).)

\[
= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y) \, dy \, dx - \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) \, dy \, dx
\]
Def: The average value of \( f(x,y) \) on \( D \) is defined as:

\[
\frac{1}{\text{Area}(D)} \iint_D f(x,y) \, dA
\]
Recall changing variables in single int:

\[ \int_{a}^{b} e^{x^2+1} \, dx \quad \rightarrow \quad \frac{1}{2} \int_{a}^{b} e^{u} \, du \]

\[ \begin{cases} u = x^2 + 1 \\ du = 2x \, dx \end{cases} \]

(a.k.a. int. by substitution)

Can we do this in Double int?

\[ \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{1-y^2}} \left( \frac{1}{\sqrt{x^2 + y^2}} \right) \, dx \, dy \]

eg)

\[ \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \]

"polar coord"
Recall polar coord: $P = (r, \theta)$

Polar coord of $P$: $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(\frac{y}{x})$

Note: Some like to impose condition $r \geq 0$.

This not necessary though and we can impose no restriction on $r$, $\theta$.

\[ \begin{align*}
  r &= \frac{1}{2} \\
  \theta &= \frac{\pi}{3} \\
  x &= \frac{1}{2} \\
  y &= \frac{\sqrt{3}}{2} \\
  \end{align*} \]
Step 1: Express integrand in polar form.

\[ f(x, y) = \frac{1}{x^2 + y^2} \]

Use \( x = r \cos \theta \), \( y = r \sin \theta \).

\[ f(r, \theta) = \frac{1}{r^2} \]

Step 2: Express region of integration in polar form.

\[ 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2 \]

Now, to transform to polar coordinates:

\[ x^2 + y^2 = r^2 = 4 \]

Region of integration:

\[ x^2 + y^2 = 4 \]

\( 0 \leq \theta \leq \frac{\pi}{2} \)

\[ 0 \leq r \leq 2 \]
Step 3: (change "dx dy" in terms of polar)

at any pt. $P = (r, \theta)$ in region, suppose we:

1. move along $\vec{V}_1$ where $r$ changes by $dr$
   $\theta$ changes by $d\theta$

   \[ dx = x_r dr + x_\theta d\theta = \cos \theta dr \]
   \[ dy = y_r dr + y_\theta d\theta = \sin \theta dr \]

   \[ \Rightarrow \vec{V}_1 = \langle \cos \theta dr, \sin \theta dr \rangle \]

2. move along $\vec{V}_2$ where $r$ changes by $d\theta$
   $\theta$ changes by $d\theta$

   \[ dx = x_\theta dr - x_r d\theta = r \sin \theta d\theta \]
   \[ dy = -r \cos \theta d\theta \]

   \[ \Rightarrow \vec{V}_2 = \langle -r \sin \theta d\theta, r \cos \theta d\theta \rangle \]
Now area of parallelogram spanned by \( \overrightarrow{V_1}, \overrightarrow{V_2} \) is just

\[
dA = r \, dr \, d\theta
\]

found by calculating \( \| \overrightarrow{V_1} \times \overrightarrow{V_2} \| \)

thought of as 3-dim vectors.

Thus integral becomes

\[
\int_0^{2\pi} \int_0^{\pi} r \, \sin \theta \, dr \, d\theta = \int_0^{2\pi} \int_0^1 dr \, d\theta
\]

\[
= \int_0^{2\pi} 1 \, d\theta
\]

\[
= 0 \bigg|_0^{2\pi}
\]

\[
= 2\pi
\]
Theorem  If a region $R$ in $xy$ plane is described by:

$$a \leq \theta \leq b$$

$$g_1(\theta) \leq r \leq g_2(\theta)$$

Then

$$\iint_R f(x,y) \, dA = \int_a^b \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) \, r \, dr \, d\theta$$
(x-1)^2 + y^2 = 1 divided into 2 regions by line y = x. Let R be bigger region.

Find volume under z = \sqrt{x^2 + y^2}, over R.

\[ (x-1)^2 + y^2 = 1 \]
\[ (1 \cos \theta - 1)^2 + r^2 \sin^2 \theta = 1 \]
\[ r^2 \cos^2 \theta - 2 \cos \theta + 1 + r^2 \sin^2 \theta = 1 \]
\[ r^2 - 2 \cos \theta = 0 \quad \text{or} \quad \Gamma = 2 \cos \theta \]

Note: need 2 integrals to find volume with cartesian coord.
\[ \iint_{R} \sqrt{x^2 + y^2} \, dA = \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos \theta} \sqrt{r^2} \, r \, dr \, d\theta \]

\[ = \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos \theta} \frac{r^3}{3} \, dr \, d\theta \]

\[ = \int_{-\pi/2}^{\pi/2} \frac{8 \cos^3 \theta}{3} \, d\theta \]

\[ = \left[ \frac{8 \sin \theta - 8 \sin^3 \theta}{3} \right]_{\theta = -\pi/2}^{\theta = \pi/2} \]

\[ = - - - \text{YOU DO THIS NOW} \]