Lect 15

ex) Find equation of tangent plane to surface \(2x^2 + y^2z = 3\) at pt \((1, 1, 1)\).

(Really mean \(z = f(x, y)\) implicit defined, and finding tangent plane to graph at \((1, 1, 1)\))

A: Must find \(Z_x, Z_y\). Can use formulae or "just do it" as:
\( \frac{x}{y} = \frac{1}{y} \)
Retelling the chain rule story...

Given \( z = f(x, y) \) and \( x(t) = x_0 \); \( x'(t) = a \) then
\[
y(t) = y_0 \quad y'(t) = b
\]

Chain rule says:
\[
\frac{dz}{dt} = f_x(p) \cdot a + f_y(p) \cdot b \quad (p = (x_0, y_0))
\]

We are going to re-express this with new notation and terminology.
**Def**

Given \( f(x,y) \) a pt. \( p \) in domain, and a vector \( \vec{V} = \langle a, b \rangle \)

1. **Define gradient vector** of \( f \) at \( p \) as:
   \[
   \nabla f(p) := \langle f_x(p), f_y(p) \rangle
   \]

2. **Define directional derivative** of \( f \) at \( p \) in dir. \( \vec{V} \) as
   \[
   D_{\vec{V}}f(p) := \nabla f(p) \cdot \vec{V} = f_x(p) \cdot a + f_y(p) \cdot b
   \]

* \( D_{\vec{V}}f(p) \) is rate of change of \( f \) at \( p \) in dir. \( \vec{V} \). (Note formula is just same as from chain rule!)

* Customary to require \( \vec{V} \) above to be unit vector to ensure fair comparison of different directions at \( p \)
Let \( f(x, y) = x^2 + y^3 \), \( p = (1, 1) \)

i) \( \nabla f\bigg|_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})} = \langle 2, 3 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, 1 \rangle = \frac{5}{\sqrt{2}} \)

ii) \( \nabla f\bigg|_{\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle} = \langle 2, 3 \rangle \cdot \frac{1}{\sqrt{5}} \langle 2, 1 \rangle = \frac{7}{\sqrt{5}} \)

iii) Find \( \mathbf{v} \) (unit) so that \( \nabla f(p) = \mathbf{0} \)

\( \mathbf{v} = \langle a, b \rangle \) want \( \langle 2, 3 \rangle \cdot \langle a, b \rangle = 2a + 3b = 0 \)

\( \Rightarrow \mathbf{v} = \langle 3, 2 \rangle \) works! \( \frac{\mathbf{v}}{\sqrt{13}} \)

iv) Find \( \mathbf{v} \) (unit) so that \( \nabla f(p) \) is maximal

\( \mathbf{v} = \langle a, b \rangle \) \( \| \mathbf{v} \| = 1 \)

Want to make \( \langle 2, 3 \rangle \cdot \langle a, b \rangle \) as large as possible where \( a^2 + b^2 = 1 \)
\[ \nabla f(p) \cdot \vec{v} \]
\[ = \cos \theta \left( \| \nabla f(p) \| \cdot \| \vec{v} \| \right) \]
\[ = \cos \theta \| \nabla f(p) \| \]

maximized when \( \cos \theta = 1 \)

1. \( \theta = 0 \)
2. \( \theta = \pi \)
3. \( \theta = \frac{\pi}{2} \)

\[ \vec{v} = \frac{\nabla f(p)}{\| \nabla f(p) \|} \]

\( \vec{v} = \left\{ \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\} \)
More on gradient vectors:

Theorem

Suppose \( \nabla f(p) \neq \mathbf{0} \). Then

1. \( \nabla f(p) \) is maximal (overall unit \( \mathbf{v} \)) when \( \mathbf{v} = \frac{\nabla f(p)}{\| \nabla f(p) \|} \)

and maximum value is \( \| \nabla f(p) \| \).

2. \( \nabla f(p) \) minimal

minimum value is \( \frac{\nabla f(p)}{\| \nabla f(p) \|} \)

\( \mathbf{v} = -\frac{\nabla f(p)}{\| \nabla f(p) \|} \).
2. \[ \nabla f(p) \text{ perpendicular to level curve } f(x,y)=f(p) \text{ at } p \]

OR

\[ \nabla f(p) \text{ normal to level surface } f(x,y,z)=f(p) \text{ at } p \]

In other words, tangent plane to a surface \[ F(x,y,z)=c \] (c constant) at \( P=(a,b,c) \) is

\[ f_x(p)(x-a)+f_y(p)(y-b)+f_z(p)(z-c)=0 \]
Thus both \( df \) and \( \text{length of } \partial f \) have important meaning rel. to \( f \).

We have already proved (1).

We can see (2) as follows (illustr.)

Let \( V \) be in tangent plane to level surface \( \partial F \) at \( P \).

\( \nabla F(P) \)
• starting at \( p \), then moving along \( \vec{V} \) is like moving on Surface.

\[ \Rightarrow \nabla F(p) \cdot \vec{V} = 0 \]

\[ \Rightarrow D_{\vec{V}} F(p) = 0 \]

• \( \nabla F(p) \) is normal for tangent plane!

Note: this gives now 2 ways of finding tangent plane to \( F(x,y,z) = c \) at \( P(a,b,c) \)
Mind I: use implicit diff to find 
\[ Z_x(\mathbf{a}, b), Z_y(\mathbf{a}, b) \] 
then use formula 
\[ Z = C + Z_x(\mathbf{a}, b)(x-a) + Z_y(\mathbf{a}, b)(y-b) \]

Mind 2: just use formula 

\[ F_x(\mathbf{a}) (x-a) + F_y(\mathbf{a}) (y-b) + F_z(\mathbf{a}) (z-c) = 0 \]

Boole talks about following:

- Normal line to a surface at \( \mathbf{a} \) at \( \mathbf{P} \).
- Line through \( \mathbf{P} \), with \( \mathbf{P} \) normal to tangent plane.

Much Easier:
equation of tangent plane to

\[ F(x, y, z) = \overbrace{z^3 x + z y + y^2}^{= 3} \]

at pt. (1, 1, 1)?

**Method A**

find \( \frac{\partial F}{\partial x}(1,1) \) and \( \frac{\partial F}{\partial y}(1,1) \) implicitly

Then use earlier formula for tangent plane
to graph

**Method B**

use formula we just learned

\[
\begin{align*}
F_x &= z^3 x = 1 \\
F_y &= z + 2y = 3 \\
F_z &= 3z^2 x + y = 4
\end{align*}
\]

= plane \( \overbrace{3}^{\circ} \)

\[ 1 \cdot (x - 1) + 3 \cdot (y - 1) + 4 \cdot (z - 1) = 0 \]
(1) equation of tangent plane to
\[ z^2x + zy + y^2 = 3 \]
at \((1,1,1)\)?

\[ F(x,y,z) \]

\[
\begin{align*}
F_x &= z^2 = 1 \\
F_y &= z + 2y = 3 \\
F_z &= 3z^2x + y = 4
\end{align*}
\]

\[ (x-1) + 3(y-1) + 4(z-1) = 0 \] plane!
(Ex) on a past final, not yet posted.

Level curves of some f(x,y).
Sketch \( \nabla f \) at \( P, Q, R \).

Note: lengths of \( \nabla f \) above are different! (why?)
i) given at \( P(2,1,1) \):
\[
\begin{align*}
T(P) &= 5 \\
T_x(P) &= 1; T_y(P) = 2; T_z(P) = 3
\end{align*}
\]

Bee starts at \( P \), flies along unit vector towards \( Q(3,2,2) \). What rate of change of Temp does Bee experience?

ii) Let \( S(x,y,z) = x + 2z \).
Bee wants to fly along dir. which rate of change of \( T, S \) both zero. Find this dir. (unit)
\[ D_{\mathbf{T}}(\mathbf{p}) = \mathbf{T}(\mathbf{p}) \cdot \left< \frac{1, 1, 1}{\sqrt{3}} \right> \]

\[ = \frac{1+2+3}{\sqrt{3}} \]

\[ = \frac{6}{\sqrt{3}} \]

\[ \mathbf{T}(\mathbf{p}) = \left\{ \begin{array}{l}
\mathbf{D}_{\mathbf{T}}(\mathbf{p}) = 0 \\
\mathbf{D}_{\mathbf{S}}(\mathbf{p}) = 0
\end{array} \right. \Rightarrow \left\{ \begin{array}{l}
a + 2b + 3c = 0 \\
a + c = 0
\end{array} \right. \]

\[ \mathbf{v} = \left< a, b, c \right> \]

\[ \Rightarrow a = t \Rightarrow c = -t \]

\[ b = t \]

So \[ \mathbf{v} = \left< 1, 1, -1 \right> \text{ or } \left< -1, -1, 1 \right> \]

both work!

If asked for unit vector, give \[ \left< 1, 1, -1 \right>/\sqrt{3} \]
2014 WTI 267 -- read problem --

\[ P = (0,4,1) \]

\[ \nabla F(p) \]

\[ \nabla T(p) \]

Temperature vspace

\[ T(x,y,z) = 5 + xy - z^2 \]

Surface: \[ z^2 + xz + y^2 = 2 \]

ant wants to move in tangent plane to increase temp. most rapidly. find d\sigma !

(would be like dotted arrow in pic)

\[ \text{it is the vector } \vec{\imath} \text{ in diagram !} \]
look! See!

\[ \vec{v} = \nabla T(p) - \text{proj}_{\nabla F(p)} \nabla F(p) \]

Now just find \( \nabla T, \nabla F \) at \( p \),
and can divide above by it's length
if you want a unit vector.
Maxima & Minima

This section only deals with 2-variable functions $f(x,y)$. The main question here:

(l) Finding relative maxima, minima of $f(x,y)$:

We say that $f(x,y)$ has

- a relative max at $(a,b)$ if $f(a,b) > f(x,y)$ for all $(x,y)$ in some disc around $(a,b)$
- a relative min if $f(a,b) < f(x,y)$

$(a,b)$
a saddle at \((a,b)\) if it has neither rel max or min there.

\[ z = f(x,y) \]

- f has rel max at P
- Saddle at R
- rel min at Q
- rel max at O
to find where \( f(x,y) \) has local (rel) extrema we apply

\[ \text{1st Derivative Test:} \]

If \( f(x,y) \) has a rel max or min at \((a,b)\),
Then either

i) \[
\begin{align*}
    f_x(a,b) &= 0 \\
    f_y(a,b) &= 0
\end{align*}
\]

or

ii) one of \( f_x(a,b) \), \( f_y(a,b) \) undefined

\((ab)\) is called critical pt. if i) or ii) hold

\[ * \] test would have located \( \Theta, P, Q \) in pic.

but also R, which is not max/min.
2nd Derivative Test:

Assume \( f(x,y) \) has critical pt. at \((a,b)=P\).
Assume 2nd partials of \( f \) continuous around \( P \).

Let 
\[
D(x,y) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (f_{xx}f_{yy} - f_{xy}^2)
\]

1. \( D(P) > 0 \) ; \( f_{xx}(P) > 0 \) \( \Rightarrow \) \( f \) has rel min at \( P \)
2. \( D(P) > 0 \) ; \( f_{xx}(P) < 0 \) \( \Rightarrow \) \( f \) has rel max at \( P \)
3. \( D(P) < 0 \) \( \Rightarrow \) \( f \) has saddle at \( P \)

(Test inconclusive if \( D(P) = 0 \))

* Note consistency of test with the model cases

\( f(x,y) = -x^2 - y^2 \) ; \( D(x,y) = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} \)
\( f(x,y) = x^2 + y^2 \) ; \( D(x,y) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \)
\( f(x,y) = -x^2 + y^2 \) ; \( D(x,y) = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} \)
ex) Classify critical pts of

\[ f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 4 \]

(ie: Find rel max & mins)

1. Find critical pts.

\[
\begin{align*}
  f_x &= 6xy - 6x = 0 \\
  f_y &= 3x^2 + 3y^2 - 6y = 0
\end{align*}
\]

\[ 6x(y - 1) = 0 \]

\[ x = 0 \quad \text{or} \quad y = 1 \]

\[ \begin{array}{c}
  x = 0 \\
  \rightarrow 3y^2 - 6y = 0 \\
  \rightarrow 3y(y - 2) = 0 \\
  y = 0, 2
\end{array} \]

\[ \begin{array}{c}
  y = 1 \\
  \rightarrow 3x^2 + 3 - 6 = 0 \\
  \rightarrow x^2 = 1 = 0 \\
  \therefore \ x = \pm 1
\end{array} \]

4 critical pts

\[ (1, 0), (0, 2), (-1, 0), (1, 1) \]
2. Use 2nd Der. Test

\[ D = f_{xx} f_{yy} - f_{xy}^2 \]
\[ = 6^2(y-1)^2 - (6x)^2 \]

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<thead>
<tr>
<th>( (0,0) )</th>
<th>( (0,2) )</th>
<th>( (1,1) )</th>
<th>( (-1,1) )</th>
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<td>( D )</td>
<td>36 &gt; 0</td>
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<tr>
<td>( f_{xx} )</td>
<td>-6 &lt; 0</td>
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Real max at \( (0,0) \)

You do these ones!
7. Classify the critical points of

\[ f(x, y) = x^2 + xy - 3y \]

\[ f_x(x, y) = 2x + y - 3 \quad f_y(x, y) = x + 2y \]

\[ f_{xx}(x, y) = 2, \quad f_{yy}(x, y) = 2 \]

(Note: \( f_{xy} \) is always defined.)

Now, could factor the top line: \( y(2x + y - 3) = 0 \)

or \( 2x + y - 3 = 0 \)

so instead of solve...
alternately

\[ x^2 - 3x = y^2 - 3y \]

\[ (x - \frac{3}{2})^2 + \frac{9}{4} = (y - \frac{3}{2})^2 + \frac{9}{4} \]

\[ x = y \quad \text{or} \quad x = -y + 3 \]

\[ 2x^2 + x^2 - 3x = 0 \]

\[ 3x^2 - 3x = 0 \]

\[ x(x-1) = 0 \]

\[ x = 0, 1 \]

\[ (0,0), (1,1) \]

4 critical pt

\( (0,3), (3,0) \)

(2) Now use 2nd Der. Test.
Why does 2nd Der. Test work?

Let \( f(x, y) \) have critical pt at \( P \).
Let \( C \) be a curve on graph of \( f \) through \((p, f(p))\). Let \( \vec{u} = \langle a, b \rangle \) be dir. vector of line "below \( C \).

1. \( \nabla \vec{u} \cdot \nabla f(p) = f_x(p) \cdot a + f_y(p) \cdot b = 0 \)

2. \( \nabla \vec{u} \cdot (\nabla f(p)) = \nabla \vec{u} \cdot (f_x \cdot a + f_y \cdot b) (p) \)

C has slope 0 over \( P \).
\[(D_u f)_a + (D_u f)_b = (f_{xx} a + f_{xy} b) a + (f_{xy} a + f_{yy} b) b = f_{xx} (a + f_{xy} b)^2 + b^2 \left( f_{xx} \left( f_{xy} f_{yy} - f_{xxy} \right) \right) \]

\[ \text{In case 2 of 2nd Der. Test, above is } < 0 \text{ at } p \]

\[ (D_u (D_u f))(p) < 0 \]

\[ \Rightarrow \text{C has local max at } p \]

\[ \Rightarrow \text{since } C \text{ was arbitrary, } f \text{ has local max at } p \]
(i) Maximize Vol of open-top box

make from 12 m² 86 card board.

\[2xz + 2yz + xy = 12\]

\[x, y > 0\]

\[\text{Vol} = xy^2\]

\[\text{eliminate } z\]

\[z = \frac{12 - xy}{2} \frac{2x^2y^2}{2(x+y)}\]

\[V(x,y) = \frac{12xy - x^2y}{2(x+y)}\]

\[\text{Maximize } V(x,y) \text{ over all } (x,y) \text{ so that } x, y > 0\]
Now we know $V(x, y)$ attains a max value in region above (by intuition and nature of problem). Thus this happens at a critical point! So we just need to locate critical points then compare $V$ at these...

---

$$\begin{align*}
V_x &= \frac{y^2 (12 - 2xy - x^2)}{2(x+y)^2} \\
V_y &= \frac{x^2 (12 - 2xy - y^2)}{2(x+y)^2}
\end{align*}$$
\[
\begin{cases}
(12 - 2x) - (x^2 - 2) = 0 \\
(12 - 2y + 4) - (x^2 - 2) = 0
\end{cases}
\]

\[0 \geq x, \quad \text{and} \quad 0 \leq x \leq 12, \quad \text{and} \quad 0 \leq y \leq 12 \]

\[x^2 = y \quad \Rightarrow \quad x = y \]

Substitute, get

\[x = 2, \quad y = 2, \quad z = 1\]

Max U(x) is 2.2.1

This was only possible since we knew a Max

If \( U \) was obtained at a critical point, before hand!
Finding Max, Min of \( f(x,y) \) on closed & bounded sets

In prev. example, we knew (by geometry + intuition) that \( f(x,y) \) attained a max value on the set \( \partial D \) of \( x \geq y \) in the positive quadrant.

Q: given \( f(x,y) \) and some set \( D \) in \( \mathbb{R}^2 \), does \( f \) always attain a max, min on \( D \)?

A: not always! Even in prev. example, \( f \) does not attain a min in the positive quadrant under right conditions though, answer to Q is YES!
Theorem (Extreme Value Theorem)

Let \( f(x,y) \) be continuous on a closed and bounded domain \( D \subset \mathbb{R}^2 \). Then \( f \) attains a max and min value on \( D \).

* eg) Some closed & bounded sets:

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- Closed: contains boundary
- Bounded: contained in some disc
Thus we can locate max say, \( f(x,y) \) on closed bdd \( D \) by:

1. locate critical pts inside \( D \) and evaluate \( f \) at these pts.

2. determine max, min value of \( f \) on boundary of \( D \).

Then max of \( f \) on \( D \) is max of the values from 1, 2.

(think why)

Similar procedure for finding min of \( f \) on \( D \).