The University of British Columbia
MATH 200 Final Examination - Dec 12, 2016

Closed book examination Time: 150 minutes

Special Instructions:

No memory aids, calculators, or electronic devices of any kind are allowed on the test. Where blanks are provided for answers, put your final answers in them. UNLESS OTHERWISE SPECIFIED, SHOW ALL YOUR WORK; little or no credit will be given for answers without the correct accompanying work. Numerical answers should be left in calculator-ready form, unless otherwise indicated. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

• Each candidate must be prepared to produce, upon request, a UBCcard for identification.
• Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
• Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
• Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

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Page 1 of 12 pages
1. Let \( A = (0, 2, 2), B = (2, 2, 2), C = (5, 2, 1) \)

3pts (a) The line which contains \( A \) and is perpendicular to the triangle \( ABC \) has parametric equations:

\[
\begin{align*}
\begin{cases}
  x = 0 \\
  y = 2 + 2t \\
  z = 2 \\
\end{cases}
\end{align*}
\]

\[
\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix}
  2 \\
  0 \\
  0 \\
\end{bmatrix} = \langle 0, 2, 0 \rangle
\]

3pts (b) The set of all points \( P \) such that \( \overrightarrow{PA} \) is perpendicular to \( \overrightarrow{PB} \) form a Plane/ Line/ Sphere/ Cone/ Paraboloid/ Hyperboloid (circle one) in space which satisfies the equation:

\[
(x-1)^2 + (y-2)^2 + (z-2)^2 = 1
\]

\[
\langle x, y-2, z-2 \rangle \cdot \langle x-2, y-2, z-2 \rangle = 0
\]

\[
x(x-2) + (y-2)^2 + (z-2)^2 = 0
\]

\[
= x^2 - 2x
\]

\[
= (x-1)^2 - 1
\]

3pts (c) If a light source at the origin shines on triangle \( ABC \) making a shadow on the plane \( x + 7y + z = 32 \) (see diagram), then \( \tilde{A} = (0, \frac{4}{7}, \frac{4}{7}) \).
2a) Some level curves of a function $f(x, y)$ are plotted in the $xy$ plane below. For each of the four statements below, circle the letters of all points in the diagram where the situation applies. For example, if the statement were "These points are on the $y$-axis" you would circle both $P$ and $U$, but none of the other letters. You may assume that a local maximum occurs at Point $T$.

i) gradient $f$ is zero

ii) $f$ has a saddle point

iii) the partial derivative $f_y$ is positive

iv) the directional derivative of $f$ in the direction $<0, -1>$ is negative.

2b) The diagram below shows three "$y$ traces" of a graph $z = F(x, y)$ plotted on $xz$ axes. (namely, the intersections of the surface $z = F(x, y)$ with the three planes $y = 1.9, y = 2, y = 2.1$.) For each statement below, circle the correct form.

i) The first order partial derivative $F_x (1, 2)$ is positive/negative/zero (circle one)

ii) $F$ has a critical point at $(2, 2)$

iii) The second order partial derivative $F_{xy} (1, 2)$ is positive/negative/zero (circle one)
3. Consider the functions \( F(x, y, z) = x^3 + xy^2 + xz \) and \( G(x, y, z) = 3x - y + 4z \). You are standing at the point \( P(0, 1, 2) \).

5pts (a) If you jump from \( P \) to \( Q(0.1, 0.9, 1.8) \) then the amount by which \( F \) changes is approximately:

\[
0.3 - 2.4 \quad \text{(use linear approximation)}.
\]

\[
\begin{aligned}
F_x &= y^2 + z = 3 \\
F_y &= 2xy = 0 \quad \text{at } P \\
F_z &= 3z^2 + x = 12
\end{aligned}
\]

\[
\therefore \Delta F = 3(0.1) + 0 + 12(-.2)
\]

3pts (b) If you jump from \( P \) in the direction along which \( G \) increases most rapidly, then \( F \) will increase/decrease (circle one and explain below).

\[
\nabla G = \langle 3, -1, 4 \rangle \quad ; \quad \nabla F(P) \cdot \nabla G(P) = 3 \cdot 3 + 4 \cdot 12 > 0
\]

3pts (c) You jump from \( P \) in a direction \( (a, b, c) \) along which rate of change of \( F \) and \( G \) are both zero. An example of such a direction is \( (a, b, c) = \langle -4, -8, 1 \rangle \) (need not be unit vector).

\[
\begin{aligned}
\nabla F(P) \cdot \vec{v} &= 3a + 12c = 0 \\
\nabla G(P) \cdot \vec{v} &= 3a - b + 4c = 0
\end{aligned}
\]

Let \( c = 1 \) \( \Rightarrow a = -12/3 = -4 \)

\[
\Rightarrow b = 3(-4) + 4(-1) \Rightarrow b = -8
\]
4. Suppose \( f(x, y) \) is twice differentiable (with \( f_{xy} = f_{yx} \)), and \( x = r \cos \theta \) and \( y = r \sin \theta \).

7pts (a) Fill blanks below in terms of functions depending on \( r \) and/or \( \theta \), and partial derivatives of \( f \) with respect to \( x \), \( y \).

\[
\begin{align*}
    f_\theta &= f_x (r \cos \theta) + f_y (r \sin \theta) \\
    f_r &= f_x (\cos \theta) + f_y (\sin \theta) \\
    f_{r\theta} &= -r \sin \theta f_x + r \cos \theta f_y + \frac{r f_{xx}}{r} + \frac{r \cos \theta}{r} f_{x\theta} + \frac{r^2 \cos^2 \theta}{r} - f_{xx} f_{yy} + \frac{f_{yy} + \sin \theta}{r} f_{y\theta} \\
\end{align*}
\]

\[
\begin{align*}
    f_\theta &= f_x (r \cos \theta) + f_y (r \sin \theta) \\
    f_r &= - \quad - \quad - \\
\end{align*}
\]

\[
\begin{align*}
    f_{r\theta} &= \frac{f_{xx} (\cos \theta) + f_{xy} (\sin \theta)}{\downarrow} \\
    &\quad \downarrow \\
    &\quad \left[ f_{xx} (\cos \theta) + f_{xy} (\sin \theta) \right] \\
    &\quad \left[ f_{yy} (\cos \theta) + f_{y\theta} (\sin \theta) \right] \\
\end{align*}
\]

2pts (b) Let \( g(x, y) \) be another function satisfying \( g_x = f_y; g_y = -f_x \). Fill blanks below with constants or functions depending on \( r \) and/or \( \theta \)

\[
\begin{align*}
    f_r &= \frac{r}{r} g_\theta \\
    f_\theta &= -\sqrt{r} \quad g_r \\
\end{align*}
\]

\[
\begin{align*}
\text{from above:} \\
\begin{cases}
    f_r = -g_y (r \sin \theta) + g_x (r \cos \theta) \\
    f_\theta = -g_y (r \sin \theta) + g_x (r \cos \theta) \\
\end{cases}
\end{align*}
\]
5. The temperature in the plane is given by \( T(x, y) = e^y(x^2 + y^2) \).

(a) 3pts (i) To find the warmest and coolest points on the circle \( x^2 + y^2 = 100 \) using Lagrange multipliers, we must solve the following system:

\[
\begin{align*}
\frac{e^y 2x}{x^2 + y^2} &= 2x \quad \text{(1)} \\
\frac{e^y(x^2 + y^2) + e^y 2y}{x^2 + y^2} &= 2y \quad \text{(2)} \\
x^2 + y^2 &= 100 \quad \text{(3)}
\end{align*}
\]

4pts (ii) By solving the above system, we conclude the warmest point on the circle is: \((0, 10)\), and the coolest point is: \((0, -10)\).

\[
\begin{align*}
2x & = \frac{x 2x}{x^2 + y^2} \\
2y & = \frac{x^3 + xy^2 + 2xy}{x^2 + y^2}
\end{align*}
\]

\(\Rightarrow 2x = x^3 + xy^2 + 2xy\)

\(\Rightarrow x(x^2 + y^2) = 0\)

\(\Rightarrow x = 0, y = \pm 10\)

\((0, \pm 10)\)

\[
\begin{align*}
T(0, 10) &= e^{10} \cdot 100 \\
T(0, -10) &= e^{-10} \cdot 100
\end{align*}
\]

Check: \(\lambda \neq 0, y \neq 0\)

\(x^2 + y^2 + 2y \neq 0\)

\(y = 0\)

\(\Rightarrow x = \pm 10\)

\(\Rightarrow e^y 100 = 0\)

\(\Rightarrow\) can't happen

\(\lambda = 0\)

\(\Rightarrow x = 0\)

\(\Rightarrow y = \pm 10\)

\(\Rightarrow e^y 100 + e^2(-10) = 0\)

\(\Rightarrow\) can't happen

\(x^2 + y^2 + 2y = 0\)

\(\Rightarrow 100 + 2y = 0\)

\(\Rightarrow y = -50\)

\(\Rightarrow x^2 + y^2 \geq 50^2\)

\(\Rightarrow\) can't happen
(b) Take the same temperature function as in part (a) \( T(x, y) = e^y(x^2 + y^2) \).

1pts (i) To find the critical points of \( T(x, y) \) we must solve the following system:

\[
\begin{align*}
2e^y x &= 0 \\
e^y (x^2 + y^2) + e^y 2y &= 0
\end{align*}
\]

3pts (ii) By solving the above system we conclude the critical points are: \( (0, 0), (0, -2) \)

\[ x = 0 \] \[ \Rightarrow x = 0 \] \[ y^2 + 2y = 0 \]

\[ y(y + 2) = 0 \]

\[ y = 0, -2 \]

\[
\begin{align*}
T(0, 0) &= 1 \\
T(0, -2) &= e^{-2} \cdot 4
\end{align*}
\]

\text{Note:} \ (e^{-2} \cdot 4 > e^{-10} \cdot 100)

2pts (c) The coolest point on the solid disc \( x^2 + y^2 \leq 100 \) is \( (0, -10) \)
6. Let \( I = \int_{0}^{1} \int_{0}^{x^2} x^3 \sin (y^3) \, dy \, dx \)

2pts (a) Sketch the corresponding region of integration in the xy plane (label your sketch sufficiently so that one could conversely use it to determine the limits of double integration)

5pts (a) Evaluate \( I \).

SEE LECTURE NOTES FOR MULTIPLE INT. PROBLEM SOLUTIONS
7. Let \( S \) be the region in the first octant (so \( x, y, z \geq 0 \)) which lies above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \((z-1)^2 + x^2 + y^2 = 1\). Let \( V \) be its volume.

3pts (a) Fill in the blanks below using cylindrical coordinates (no explanation required)

\[
V = \iiint (\text{blank}) \, r \, dr \, d\theta \, dz
\]

3pts (b) Fill in the blanks below using spherical coordinates (no explanation required)

\[
V = \iiint (\text{blank}) \, d\rho \, d\phi \, d\theta
\]

4pts (c) Calculate \( V \) using either (not both) of the integrals above.
8. [11pts] Let $E$ be the region bounded by the planes $y = 0, y = 2, y + z = 3$ and the surface $z = x^2$. Consider the integral

$$I = \int \int \int_E f(x, y, z) dV.$$

Fill in the blanks below (No explanations required. Also, in each part below, you may only need one integral to express your answer, in which case leave the other blank)

(a) $I = \int \int \int f(x, y, z) dz dx dy + \int \int \int f(x, y, z) dx dy dz$

(b) $I = \int \int \int f(x, y, z) dx dy dz + \int \int \int f(x, y, z) dx dy dz$

(c) $I = \int \int \int f(x, y, z) dy dx dz + \int \int \int f(x, y, z) dy dx dz$

The End
\begin{align*}
\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle &= a_1 b_1 + a_2 b_2 + a_3 b_3 = |a| |b| \cos \theta \\
\text{proj}_a b &= \left( \frac{a \cdot b}{|a|^2} \right) a = \left( \frac{a \cdot b}{a \cdot a} \right) a \\
\det(a, b) &= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = -\det(b, a) \\
\det(a, b, c) &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
\det(a, b, c) &= a \cdot \left( \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \right) = a \cdot (b \times c). \\
b \times c &= \left( \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \right) \\
|b \times c| &= |b| |c| |\sin \theta| \\
\det(a, b, c) &= -\det(b, a, c) \quad \text{so} \quad a \cdot (b \times c) = -b \cdot (a \times c) \\
x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \\
r_0 + tv, \quad (1-t)r_0 + tr_1 \\
n \cdot (r - r_0) = 0, \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \\
dist(\langle ax + by + cz + d_1 = 0, ax + by + cz + d_2 = 0 \rangle = \frac{|d_1 - d_2|}{\langle a, b, c \rangle} \\
\text{Distance from a point } P(x_1, y_1, z_1) \text{ to the plane } ax + by + cz + d = 0 \text{ is} \\
D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.
\end{align*}
\[
1/2 = \sin(30^\circ) = \sin(\pi/6) = \cos(60^\circ) = \cos(\pi/3) \\
1/\sqrt{2} = \sin(45^\circ) = \sin(\pi/4) = \cos(45^\circ) = \cos(\pi/4) \\
\sqrt{3}/2 = \cos(30^\circ) = \cos(\pi/6) = \sin(60^\circ) = \sin(\pi/3)
\]

\[
\cos(2\alpha) = 1 - 2\sin^2 \alpha = 2\cos^2 \alpha - 1
\]

\[
L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\
f(x, y) - f(x_0, y_0) = z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
\Delta f = \Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y \\
df = dz = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy
\]

\[
\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}
\]

Directional derivative (\(u\) a unit vector): 
\[
D_u f = \nabla f \cdot u = \langle f_x, f_y, f_z \rangle \cdot u
\]

\[
F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0
\]

\[
f_{xx}, \quad D = f_{xx} f_{yy} - (f_{xy})^2
\]

Local min: \(f_{xx} > 0, D > 0\); local max: \(f_{xx} < 0, D > 0\); saddle: \(D < 0\); degenerate/indeterminate: \(D = 0\).

Lagrange multipliers to maximize/minimize \(f = f(x, y)\) or \(f = f(x, y, z)\) subject to \(g = C = \) Constant where \(g = g(x, y)\) or \(g = g(x, y, z)\):
\[
\nabla f = \lambda \nabla g, \quad g = C.
\]

Mass and centre of mass, density \(\rho = \rho(x, y)\):
\[
m = \int \int_D \rho(x, y) \, dA, \quad \bar{x} = (1/m) \int \int_D x \rho(x, y) \, dA, \quad \bar{y} = (1/m) \int \int_D y \rho(x, y) \, dA
\]

Polar/cylindrical coordinates: \(x = r \cos \theta, y = r \sin \theta,\)
\[
dA = dx \, dy = r \, dr \, d\theta, \quad dV = dx \, dy \, dz = r \, dx \, dy \, dz
\]

Spherical (like polar in \(x, y\) with radius \(r = \rho \sin \phi, z = \rho \cos \phi\), so \(\phi\) measures angle with positive \(z\)-axis, \(0 \leq \phi \leq \pi = 180^\circ\)):
\[
x = (\rho \sin \phi) \cos \theta, \quad y = (\rho \sin \phi) \sin \theta, \quad z = \rho \cos \phi, \quad dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]
Closed book examination

Time: 2.5 hours

Last Name __________ First __________ Signature __________

Student Number __________

Special Instructions:
No books, notes, or calculators are allowed. SHOW YOUR WORK FOR ALL PROBLEMS; NO CREDIT FOR ANSWERS WITHOUT WORK!

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<td>- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.</td>
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<td>(c) purposely viewing the written papers of other candidates;</td>
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<td>(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,</td>
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<td>(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).</td>
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<td>- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.</td>
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<td>- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.</td>
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(a) Consider the plane $4x + 2y - 4z = 3$. Find all parallel planes that are distance 2 from the above plane. Your answers should be in the following form $4x + 2y - 4z = C$.

\[
\text{Use formula sheet!}
\]

\[
\frac{|3-C|}{\sqrt{4^2+2^2+4^2}} = 2 \quad \rightarrow \quad |3-C| = 12
\]

\[
\rightarrow \quad C = -9, \ 15
\]

(b) Find the parametric equation for the line of intersection of the planes

\[\mathbf{1} \quad x+y+z = 11 \quad \text{and} \quad \mathbf{2} \quad x-y-z = 13.\]

Let $z = t$

\[
\rightarrow \quad 2x = 24 \quad (\text{eq.} \mathbf{1})
\]

\[
\rightarrow \quad x = 12
\]

\[
\rightarrow \quad y = -1 - t \quad (\text{eq.} \mathbf{2})
\]

(c) Find the tangent plane to

\[
F(x,y,z) = \frac{27}{\sqrt{x^2+y^2+z^2+3}} = 9
\]

at the point $(2, 1, 1)$.

\[
\text{Ans: } F_x(2,1,1)(x-2) + F_y(2,1,1)(y-1) + F_z(2,1,1)(z-1) = 0
\]

\[
-2(x-2) - (y-1) - (z-1) = 0
\]

\[
F_x = \frac{-27}{2(x^2+y^2+z^2+3)^{3/2}} \cdot 2x = \frac{-27}{2 \cdot 9^{3/2}} \cdot 2(2) = -2
\]

\[
F_y = \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) \cdot \begin{array}{c}
y \\
\vdots \\
\vdots \\
\end{array} = -1
\]

\[
F_z = \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) \cdot \begin{array}{c}
z \\
\vdots \\
\vdots \\
\end{array} = -1
\]
2. A function $T(x, y, z)$ at $P = (2, 1, 1)$ is known to have $T(P) = 5$, $T_x(P) = 1$, $T_y(P) = 2$, and $T_z(P) = 3$.

(i) A bee starts flying at $P$ and flies along the unit vector pointing towards the point $Q = (3, 2, 2)$. What is the rate of change of $T(x, y, z)$ in this direction?

$$\frac{\overrightarrow{PQ}}{||\overrightarrow{PQ}||} = \frac{<1, 1, 1>}{\sqrt{3}}$$

$$\nabla T(P) \cdot U = \frac{<1, 2, 3> \cdot <1, 1, 1>}{\sqrt{3}} = \frac{6}{\sqrt{3}}$$

(ii) Use linear approximation of $T$ at the point $P$ to approximate $T(1.9, 1, 1.2)$.

$$T(1.9, 1, 1.2) \approx T(2, 1, 1) + 1(-.1) + 2(0) + 3(.2)$$

$$= 5 - .1 + .6$$

$$= 5.5$$

(iii) Let $S(x, y, z) = x + z$. A bee starts flying at $P$; along which unit vector direction should the bee fly so that the rate of change of $T(x, y, z)$ and of $S(x, y, z)$ are both zero in this direction?

See 2016 #3c)
3. Let $w(s, t) = u(2s + 3t, 3s - 2t)$ for some twice differentiable function $u = u(x, y)$.

(a) Find $w_{ss}$ in terms of $u_{xx}$, $u_{xy}$, and $u_{yy}$ (you can assume that $u_{xy} = u_{yx}$).

\[
w_y = w_y \cdot x_s + w_y \cdot y_s = w_x \cdot 2 + w_y \cdot 3
\]

\[
w_{ss} = w_{xss} + w_{ys} \cdot 3 = (w_{xxx} x_s + w_{xxy} y_s) \cdot 2 + (w_{yx} x_s + w_{yy} y_s) \cdot 3
\]

\[
= w_{xxx} \cdot 2^2 + w_{xxy} \cdot 2 + 2w_{yy} \cdot 6 + w_{yy} \cdot 9
\]

(b) Suppose $u_{xx} + u_{yy} = 0$. For what constant $A$ will $w_{ss} = Aw_{tt}$?

As above, determine that

\[
w_{tt} = w_{xxx} \cdot 9 - 2w_{xxy} \cdot 6 + w_{yy} \cdot 4
\]

\[
\Rightarrow w_{ss} + w_{tt} = 4 \left( \frac{w_{xxx} + w_{xxy}}{0} \right) + 9 \left( \frac{w_{xx} \cdot w_{yy}}{6} \right)
\]

\[
\Rightarrow \boxed{w_{ss} = -w_{tt}} \quad \text{ie)} \quad A = -1
\]
4. Find and classify the critical points of \( f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 4 \)

See class notes for this one.
[9] 5. Use Lagrange multipliers to find the minimum and maximum values of \((x + z)e^y\) subject to \(x^2 + y^2 + z^2 = 6\).

\[
\begin{align*}
\n
\begin{cases}
2f = 
2\lambda \nabla g \\
8 = 6
\end{cases}

\Rightarrow

\begin{cases}
e^y = 2\lambda x \\
(x + z)e^y = 2\lambda y \\
e^y = 2\lambda z \\
x^2 + y^2 + z^2 = 6
\end{cases}
\end{align*}
\]

Note: \(\lambda \neq 0\)

\[
\Rightarrow x = z \quad (1, 3)
\]

\[
\Rightarrow x + z = \frac{y}{x} \quad (2, 6)
\]

Note: \(x \neq 0\) (1)

\[
\Rightarrow 2x = \frac{y}{x}
\]

\[
\Rightarrow y = 2x^2
\]

\[
\Rightarrow x^2 + 4x^4 + z^2 = 6 \quad (4)
\]

\[
\Rightarrow 4x^4 + 2x^2 - \frac{6}{3} = 0
\]

\[
\Rightarrow (2x^2 + 3)(x^2 - 1) = 0
\]

\[
\Rightarrow x = \pm 1
\]

Pt's are \((\pm 1, 2, \pm 1)\)

\[
\begin{align*}
\begin{cases}
f(1, 2, 1) = 2e^2 \\
f(-1, 2, 1) = -2e^2
\end{cases}
\end{align*}
\]

Max Value: \(2e^2\)

Min Value: \(-2e^2\)
6. Consider the domain $D$ above the $x$-axis and below parabola $y = 1 - x^2$ in the $xy$-plane.

(a) Sketch $D$.

(b) Express
\[
\int \int_D f(x, y) \, dA
\]
as an iterated integral corresponding to the order $dx \, dy$. Then express this integral as an iterated integral corresponding to the order $dy \, dx$.

(c) Compute the integral in the case $f(x, y) = e^{-x^2/y^2}$.

See notes for multiple integral problems.
[9] 7. Let $E$ be the region inside the cylinder $x^2 + y^2 = 1$, below the plane $z = y$ and above the plane $z = -1$. Express the integral

$$
\int \int \int_E f(x, y, z) \, dV
$$

as three different iterated integrals corresponding to the orders of integration: (a) $dz \, dx \, dy$, (b) $dx \, dy \, dz$, and (c) $dy \, dz \, dx$. 
[9] 8. The solid $E$ is bounded below by the paraboloid $z = x^2 + y^2$ and above by the cone $z = \sqrt{x^2 + y^2}$. Let 

$$I = \iiint_E z(x^2 + y^2 + z^2) \, dV$$

a) Write $I$ in terms of cylindrical coordinates. Do not evaluate.

b) Write $I$ in terms of spherical coordinates. Do not evaluate.

c) Calculate $I$.
Formula Sheet for Math 200 Final Exam, Fall 2015

\[ \langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 = |a| |b| \cos \theta \]

\[ \text{proj}_a b = \left( \frac{a \cdot b}{|a|^2} \right) a = \left( \frac{a \cdot b}{a \cdot a} \right) a \]

\[ \text{Det}(a, b) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = -\text{Det}(b, a) \]

\[ \text{Det}(a, b, c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \]

\[ \text{Det}(a, b, c) = a \cdot \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - b \cdot \begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} + c \cdot \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} = a \cdot (b \times c). \]

\[ b \times c = \begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - \begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix}, \quad |b \times c| = |b| |c| |\sin \theta| \]

\[ \text{Det}(a, b, c) = -\text{Det}(b, a, c) \quad \text{so} \quad a \cdot (b \times c) = -b \cdot (a \times c) \]

\[ x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \]

\[ r_0 + tv, \quad (1-t)r_0 + tr_1 \]

\[ n \cdot (r - r_0) = 0, \quad a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]

\[ \text{dist}(ax + by + cz + d_1 = 0, ax + by + cz + d_2 = 0) = \frac{|d_1 - d_2|}{|a|} \]

Distance from a point \( P(x_1, y_1, z_1) \) to the plane \( ax + by + cz + d = 0 \) is

\[ D = |ax_1 + by_1 + cz_1 + d| / \sqrt{a^2 + b^2 + c^2}. \]
$$\frac{1}{2} = \sin(30^\circ) = \sin(\pi/6) = \cos(60^\circ) = \cos(\pi/3)$$
$$\frac{1}{\sqrt{2}} = \sin(45^\circ) = \sin(\pi/4) = \cos(45^\circ) = \cos(\pi/4)$$
$$\sqrt{3}/2 = \cos(30^\circ) = \cos(\pi/6) = \sin(60^\circ) = \sin(\pi/3)$$

$$\cos(2\alpha) = 1 - 2\sin^2 \alpha = 2\cos^2 \alpha - 1$$

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
$$f(x, y) - f(x_0, y_0) = z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$\Delta f = \Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$
$$df = dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

Directional derivative \((u\) a unit vector):

$$D_u f = \nabla f \cdot u = \langle f_x, f_y, f_z \rangle \cdot u$$

$$f_{xx}, \quad D = f_{xx}f_{yy} - (f_{xy})^2$$

Local min: \(f_{xx} > 0, \ D > 0;\) local max: \(f_{xx} < 0, \ D > 0;\) saddle: \(D < 0;\) degenerate/indeterminate: \(D = 0.\)

Lagrange multipliers to maximize/minimize \(f = f(x, y)\) or \(f = f(x, y, z)\) subject to \(g = C =\)

Constant where \(g = g(x, y)\) or \(g = g(x, y, z)\):

$$\nabla f = \lambda \nabla g, \quad g = C.\)$$

Mass and centre of mass, density \(\rho = \rho(x, y)\):

$$m = \int \int_D \rho(x, y) \, dA, \quad \overline{x} = (1/m) \int \int_D x \rho(x, y) \, dA, \quad \overline{y} = (1/m) \int \int_D y \rho(x, y) \, dA$$

Polar/cylindrical coordinates: \(x = r \cos \theta, \ y = r \sin \theta,\)

$$dA = dx \, dy = r \, dr \, d\theta, \quad dV = dx \, dy \, dz = r \, dz \, dr \, d\theta$$

Spherical (like polar in \(x, y\) with radius \(r = \rho \sin \phi, \ z = \rho \cos \phi,\) so \(\phi\) measures angle with positive \(z\)-axis, \(0 \leq \phi \leq \pi = 180^\circ)\):

$$x = (\rho \sin \phi) \cos \theta, \quad y = (\rho \sin \phi) \sin \theta, \quad z = \rho \cos \phi, \quad dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$
The University of British Columbia
Final Examination - December 6, 2014
Mathematics 200

Closed book examination

Time: 2 hours 30 minutes

Last Name: ___________________________ First: ___________________________

Student Number: ________________________

Section (check one): □ 101 (MWF 9-10, Peterson) □ 102 (MWF 11-12, Fraser)
□ 103 (MWF 11-12, Nguyen) □ 104 (MWF 1-2, Liu)
□ 105 (TuTh 9:30-11, Roe) □ 107 (TuTh 3:30-5, Roe)

Special Instructions:
- Be sure that this examination has 13 pages. Write your name on top of each page.
- No books, notes, calculators, or any other aids are allowed.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  (a) speaking or communicating with other candidates, unless otherwise authorized;
  (b) purposely exposing written papers to the view of other candidates or imaging devices;
  (c) purposely viewing the written papers of other candidates;
  (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (Electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

| 1 | 7 |
| 2 | 10 |
| 3 | 10 |
| 4 | 8 |
| 5 | 14 |
| 6 | 14 |
| 7 | 14 |
| 8 | 14 |
| 9 | 9 |
| **Total** | **100** |
1. Suppose that \( f(x, y, z) \) is a function of three variables and let \( u = \frac{1}{\sqrt{6}}(1, 1, 2) \) and \( v = \frac{1}{\sqrt{3}}(1, -1, -1) \) and \( w = \frac{1}{\sqrt{3}}(1, 1, 1) \). Suppose that at a point \((a, b, c),\)

\[
D_u f = 0 \\
D_v f = 0 \\
D_w f = 4.
\]

Find \( \nabla f \) at \((a, b, c).\)

let \( f_x = A \) \\
\( f_y = B \) \hspace{1cm} \text{at} \ (a, b, c) \\
\( f_z = C \)

\[
\Rightarrow \begin{cases} \\
\frac{1}{\sqrt{6}} (A + B + 2C) = 0 \hspace{1cm} (1) \\
\frac{1}{\sqrt{3}} (A - B - C) = 0 \hspace{1cm} (2) \\
\frac{1}{\sqrt{3}} (A + B + C) = 4\sqrt{3} \hspace{1cm} (3) \\
\end{cases}
\]

\[
\Rightarrow 2A = 4\sqrt{3} \hspace{1cm} (2) + (3)
\]

\[
\Rightarrow A = 2\sqrt{3}
\]

\[
\Rightarrow C = -2A = -4\sqrt{3} \hspace{1cm} (1) + (2)
\]

\[
\Rightarrow B = -A - 2C \hspace{1cm} (1)
\]

\[
= -2\sqrt{3} + 8\sqrt{3} \hspace{1cm} = 6\sqrt{3}
\]

\[
\nabla f = \langle 2\sqrt{3}, 6\sqrt{3}, -4\sqrt{3} \rangle
\]
2. Let \( f(u,v) \) be a differentiable function of two variables, and let \( z \) be a differentiable function of \( x \) and \( y \) defined implicitly by \( f(xz,yz) = 0 \). Show that

\[
x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -z.
\]

\[
v = 0
\]

\[
\rightarrow f_u u_x + f_v v_x = 0
\]

\[
\rightarrow f_u (z + x z_x) + f_v (y z_y) = 0
\]

\[
\rightarrow \frac{z_x}{x f_u y f_v} = -\frac{f_u z}{x f_u y f_v}
\]

Likewise, derive that

\[
\frac{z_y}{x f_u y f_v} = -\frac{f_v z}{x f_u y f_v}
\]

\[
\frac{x z_x - y z_y}{x f_u + y f_v} = -z
\]
3. Let \( z = f(x, y) \) be given implicitly by \( e^z + yz = x + y \).

(a) Find the differential \( dz \).

\[
\begin{align*}
\text{Let } F(x, y, z) &= e^z + yz - x - y \\
\rightarrow \frac{\partial F}{\partial x} &= -\frac{F_x}{F_z} = \frac{1}{e^{x+y}} \\

\frac{\partial F}{\partial y} &= -\frac{F_y}{F_z} = \frac{1 - z}{e^{x+y}} \\

\Rightarrow dz &= \frac{1}{e^{x+y}} \, dx + \frac{(1 - z)}{e^{x+y}} \, dy
\end{align*}
\]

(b) Use linear approximation at the point \((1, 0)\) to approximate \( f(0.99, 0.01) \).

\[
f(0.99, 0.01) \approx f(1, 0) + f_x(1, 0)(-0.01) + f_y(1, 0)(0.01)
\]

\[
\begin{align*}
\text{Note:} \\
&x = 1, y = 0 \\
&\Rightarrow e^z = 1 \\
&\Rightarrow z = 0 \\
&\Rightarrow f(1, 0) = 0, f_x(1, 0) = 1, f_y(1, 0) = 1 \\
\end{align*}
\]

\[
= 0 + 1(-0.01) + 1(0.01) = 0 
\]
4. Let $S$ be the surface $z = x^2 + 2y^2 + 2y - 1$. Find all points $P(x_0, y_0, z_0)$ on $S$ with $x_0 \neq 0$ such that the normal line at $P$ contains the origin $(0, 0, 0)$.

Let $F(x, y, z) = x^2 + 2y^2 + 2y - 1 - z$

\[
\begin{align*}
F_x &= 2x \\
F_y &= 4y + 2 \\
F_z &= -1
\end{align*}
\]

The normal line at $P$ has direction vector $\langle 2x_0, 4y_0 + 2, -1 \rangle$.

The line contains origin when:

\[
\begin{align*}
0 &= x_0 + 2x_0 t \\
0 &= y_0 + (4y_0 + 2)t \\
0 &= z_0 - t \\
x_0^2 + 2y_0^2 + 2y_0 - 1 - z_0 &= 0
\end{align*}
\]

$x_0(1 + 2t) = 0$ \(\text{\textcopyright} \)

\[
\begin{align*}
\therefore x_0 &= 0 & \text{or} & (1 + 2t) &= 0 \\
& \text{can't happen} & \Rightarrow t &= -\frac{1}{2} & \Rightarrow z &= -\frac{1}{2} & \Rightarrow y &= -1 & x^2 &= z + 1 + 2\neq -2 & \text{\textcopyright}
\end{align*}
\]

\[
\therefore \text{pts are } (\pm\frac{1}{2}, -1, -\frac{1}{2})
\]
5. Let \( f(x, y) = xy(x + y - 3) \).

(a) Find all critical points of \( f \), and classify each one as a local maximum, a local minimum, or saddle point.

\[ \text{DONE IN CLASS} \]
(b) Find the location and value of the absolute maximum and minimum of $f$ on the triangular region $x \geq 0, y \geq 0, x + y \leq 8$.

\[ \text{max and min of } f(x,y) = xy(x+y-3) \text{ on triangle} \]

\[ x+y = 8 \]

also done in class notes.
6. In the $xy$-plane, the disk $x^2 + y^2 \leq 2x$ is cut into 2 pieces by the line $y = x$. Let $D$ be the larger piece.

(a) Sketch $D$ including an accurate description of the center and radius of the given disk. Then describe $D$ in polar coordinates $(r, \theta)$.

\textbf{DONE THIS AND REST IN CLASS NOTES. (except #7)}

(b) Find the volume of the solid below $z = \sqrt{x^2 + y^2}$ and above $D$. 
7. The density of hydrogen gas in a region of space is given by the formula

\[ \rho(x, y, z) = \frac{z + 2x^2}{1 + x^2 + y^2}. \]

(a) At \((1, 0, -1)\), in which direction is the density of hydrogen increasing most rapidly?

\[ \nabla \rho = \langle \rho_x, \rho_y, \rho_z \rangle = \langle 3\frac{1}{2}, 0, \frac{1}{2} \rangle \text{ at } (1, 0, -1) \]

At \((1, 0, -1)\):

\[ \rho_x = \frac{(1 + x^2 + y^2) \cdot 4x - (z + 2x^2)(2x)}{(1 + x^2 + y^2)^2} = \frac{2 \cdot 4 - 1 \cdot 1}{4} = \frac{3}{2} \]

\[ \rho_y = \frac{-(z + 2x^2)}{(1 + x^2 + y^2)^2} \cdot 2y = \frac{-1}{4} \cdot 0 = 0 \]

\[ \rho_z = \frac{1}{(1 + x^2 + y^2)} = \frac{1}{2} \]
(b) You are in a spacecraft at the origin. Suppose the spacecraft flies in the direction of \((0, 0, 1)\). It has a disc of radius 1, centred on the spacecraft and deployed perpendicular to the direction of travel, to catch hydrogen. How much hydrogen has been collected by the time that the spacecraft has traveled a distance 2? [You may use the fact that \(\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi\].

\[
\text{amt. of hydrogen collected is}
\]

\[
E : \text{cylinder} \quad z = 2
\]

\[
x^2 + y^2 = 1
\]

\[
E \quad z = 2
\]

\[
\iiint_E \rho \, d\tau = \frac{\pi^2}{2} \left[ 2 \cdot 2 \cdot \frac{\sqrt{0}}{1 + \sqrt{2}} \right]
\]

\[
= 2 \left[ \int_0^{2\pi} \frac{1}{1 + \sqrt{2}} \, d\theta + 4 \int_0^{\frac{\pi}{2}} \cos^2 \theta \left[ \frac{1}{2} - \ln \left( \frac{1 + \sqrt{2}}{2} \right) \right] \, d\theta \right]
\]

\[
= 2 \ln 2 \cdot 2 \pi + 4 \left[ \frac{\pi}{2} - \ln \left( \frac{\sqrt{2}}{2} \right) \right] \cdot 2 \pi
\]

\[
= \left( 4 \pi - 2 \pi \cdot \ln 2 \right)
\]
8. Consider the iterated integral

\[ I = \int_{-a}^{0} \int_{-\sqrt{a^2-x^2}}^{0} \int_{0}^{\sqrt{a^2-x^2-y^2}} (x^2 + y^2 + z^2)^{2014} \, dz \, dy \, dx \]

where \( a \) is a positive constant.

(a) Write \( I \) as an iterated integral in cylindrical coordinates.

(b) Write \( I \) as an iterated integral in spherical coordinates.
(c) Evaluate $I$ using whatever method you prefer.