

## § 2.3 Analytic Functions

### Lect 5

①

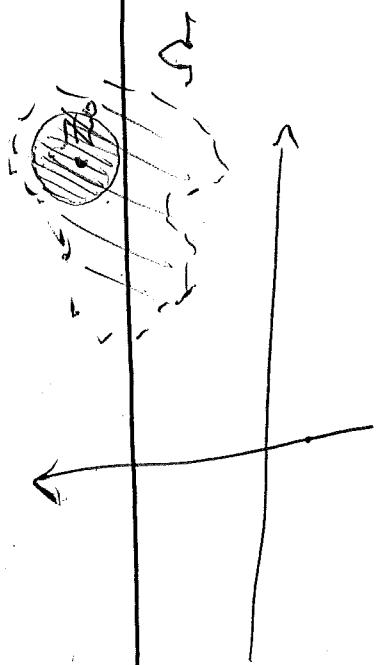
So far, we have the tools to make the:

Def

Let  $f(x+iy) = u(x,y) + i v(x,y)$  be complex function on open set  $\Omega \subseteq \mathbb{C}^n$

- ①  $f$  is called analytic at  $z_0 \in \Omega$  if in some disc  $D_s(z_0) \subset \Omega$  have:
- $$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\text{Some } a_n's \in \mathbb{C})$$

- ②  $f$  is called analytic on  $\Omega$  if it is so at each  $z_0 \in \Omega$ .



(2)

- \* Just a more precise version of our earlier start: "f(z) depends only on  $\bar{z}$ , not on  $\underline{z}$ "
- \* Polynomials  $a_n z^n + a_{n-1} z^{n-1} + \dots$  are all analytic on  $\mathbb{C}$ !
- \*  $e^z$  would be analytic if we are sure that
 
$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

}

Converges? To  $e^{\underline{z}}$ ?
- \* Is  $|z|^2 = z\bar{z}$  analytic? doesn't look so (has  $\bar{z}$ )  
but are we sure? not easy to say now.

## Def<sup>n</sup> 2

Let  $f(z)$  be complex fun on open set  $\Omega \subset \mathbb{C}$ .

①  $f(z)$  called differentiable at  $z_0 \in \Omega$  if

$$\frac{df}{dz}(z_0) := \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.

②  $f(z)$  is called analytic on  $\Omega$  if it is diff. at each  $z_0 \in \Omega$

\* Both Def<sup>n</sup> lead to same thing! (they are equivalent).  
lets see why Def<sup>n</sup> 2 may be preferable for now  
→ explained much later...

ASSUME DEF 2 FROM NOW ON

then follows from prev. example that

⑥

polynomials  $a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 s \in \mathbb{C}$   
are analytic on  $\mathbb{C}$

rational functions  $\frac{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 s}{b_m z^m + \dots + b_1 t}$   
analytic on their domain (assume  $\mathbb{B}$  an open set).

e.g. show that  $\overline{z}^n$  is not analytic (use Def 2).

$$\lim_{z \rightarrow 0} \frac{(z_0 + \Delta z)^n - \overline{z_0}}{\Delta z} = \frac{\overline{z_0}^n + \dots}{\Delta z}$$

see prev. ex)

Ex) Show  $z^n$  analytic on  $\mathbb{C}$ . ( $n \in \mathbb{Z}^+$ )

$$\begin{aligned}
 & \lim_{\Delta z \rightarrow 0} \frac{(z_0 + \Delta z)^n - z_0^n}{\Delta z} \\
 &= \lim_{\Delta z \rightarrow 0} \frac{\cancel{(z_0 + \Delta z)^n} + \binom{n}{1} z_0^{n-1} \Delta z + \binom{n}{2} z_0^{n-2} \Delta z^2 + \dots - \cancel{z_0^n}}{\Delta z} \\
 &= \lim_{\Delta z \rightarrow 0} \left( \binom{n}{1} z_0^{n-1} + \binom{n}{2} z_0^{n-2} \Delta z + \dots \right) \\
 &\qquad\qquad\qquad \downarrow \text{as } \Delta z \rightarrow 0 \\
 &= n z_0^{n-1}
 \end{aligned}$$

Shown  $z^n$  is diff at every  $z_0 \in \mathbb{C}$   
 thus analytic on  $\mathbb{C}$

$\sum (z^n)' = n z^{n-1}$

Insert here, usual rules of diff:

\*  $f, g$  diff at  $z_0$  then so are  $f+g$  at  $z_0$

$$(f+g) \text{ at } z_0$$

$$(fg) \text{ at } z_0$$

$$\left(\frac{f}{g}\right) \text{ at } z_0$$

with usual formula

$$f+g$$

$$fg$$

$$\frac{f}{g}$$

$f(g)$  at  $z_0$

and derivative is

$$f'(g) \cdot g'$$

$f$  diff at  $w_0 = f(z_0)$  then so is  $f(g)$  at  $z_0$

$g$  diff at  $z_0$

\*

(6)

$$= \lim_{\Delta z \rightarrow 0} \binom{n}{1} \overline{z_0}^{n-1} \frac{\Delta \overline{z}}{\Delta z} + \binom{n}{2} \overline{z_0}^{n-2} \frac{(\Delta \overline{z})^2}{\Delta z} + \dots$$

$$\left( \binom{n}{1} \overline{z_0}^{n-1} \frac{\Delta \overline{z}}{\Delta z} + \binom{n}{2} \overline{z_0}^{n-2} \frac{(\Delta \overline{z})^2}{\Delta z} + \dots \right)$$

↓  
↓ Verify yourself  
these terms  $\rightarrow 0$   
as  $\Delta z \rightarrow 0$ .

does not have  
limit as  $\Delta z \rightarrow 0$   
(assumed before)  
unless  $z_0 = 0$  !

$\overline{z}^n$  is differentiable at  $z_0 = 0$  only if  
This is not analytic on any open set.

(7)

Similar argument can be used to show:

any polynomial involving  $\underline{z}, \overline{z}$  (both) is not analytic anywhere.

$$\alpha_{n,m} z^n \overline{z}^m + \dots$$

eg) Show  $e^z$  is ~~not~~ analytic on  $\mathbb{C}$ !

$$\lim_{\Delta z \rightarrow 0} \frac{e^{z_0 + \Delta z} - e^{z_0}}{\Delta z} \text{ exists? (for any } z_0)$$

$$= \lim_{\Delta z \rightarrow 0} e^{z_0} \left( \frac{e^{\Delta z} - 1}{\Delta z} \right)$$

(6)

$$\text{Let } \Delta z = x + iy$$

$$= e^{\frac{z_0}{\Delta z}} \lim_{\Delta z \rightarrow 0} \left( e^{\frac{e^{x+iy} - e^0}{x+iy}} \right) \underbrace{a(z)}_{e^x \left( e^{iy} - e^0 \right) \left( \frac{iy}{x+iy} \right)}$$

$$= e^{z_0} \lim_{\Delta z \rightarrow 0} \underbrace{e^x \left( e^{iy} - e^0 \right) \left( \frac{iy}{x+iy} \right)}_{b(z)}$$

(pure algebra: add  
subtract - etc.)

$$= e^{z_0} \lim_{\Delta z \rightarrow 0} \frac{a(z) \cancel{x} + b(z) \cancel{iy}}{x+iy}$$

$$= e^{z_0} \lim_{\Delta z \rightarrow 0} a(z) + (b(z) - a(z)) \frac{iy}{x+iy}$$

(see next pg.).

Notice: as  $\Delta z \rightarrow 0$

Know

- 1)  $a(z) \rightarrow 1$
- 2)  $b(z) \rightarrow 1$

use def. of  $e^x, \cos y, \sin y$   
to see these limits.

$$3) \left| \frac{iy}{x+iy} \right| = \frac{|iy|}{\sqrt{x^2+y^2}} = \frac{\sqrt{y^2}}{\sqrt{x^2+y^2}} \leq 1$$

also

$$\therefore (b(z) - a(z)) \left( \frac{iy}{x+iy} \right) \rightarrow 0$$

$e^{az}$  is analytic on  $C$ .

$$(e^z)' = e^z$$