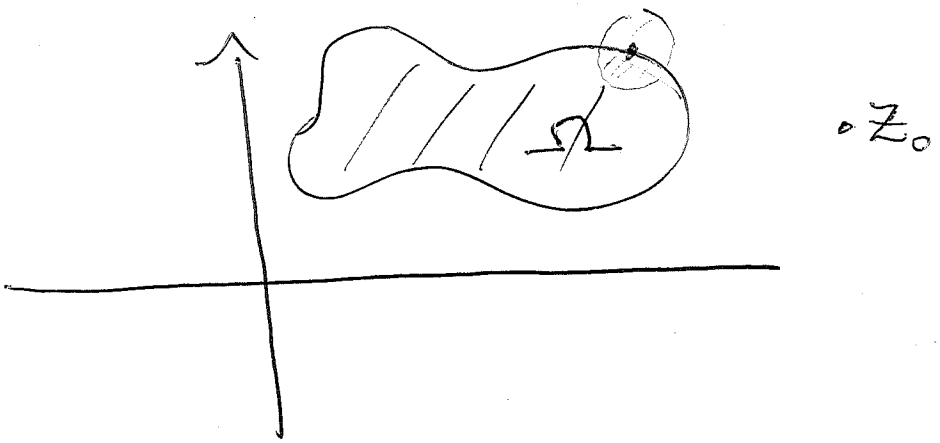
Lect 4



- z_0 interior pt. of Ω : $D_\delta(z_0) \subset \Omega$ for some $\delta > 0$
- neighborhood of z_0 : some open set containing z_0
- ~~Ω connected~~
- z_0 a boundary pt of Ω : every disc $D_\delta(z_0)$ contains pts inside Ω and outside Ω.

(Ω closed $\Leftrightarrow$ Ω contains its boundary pts).

in general, domain refers to an open + connected set

will later have to invoke:

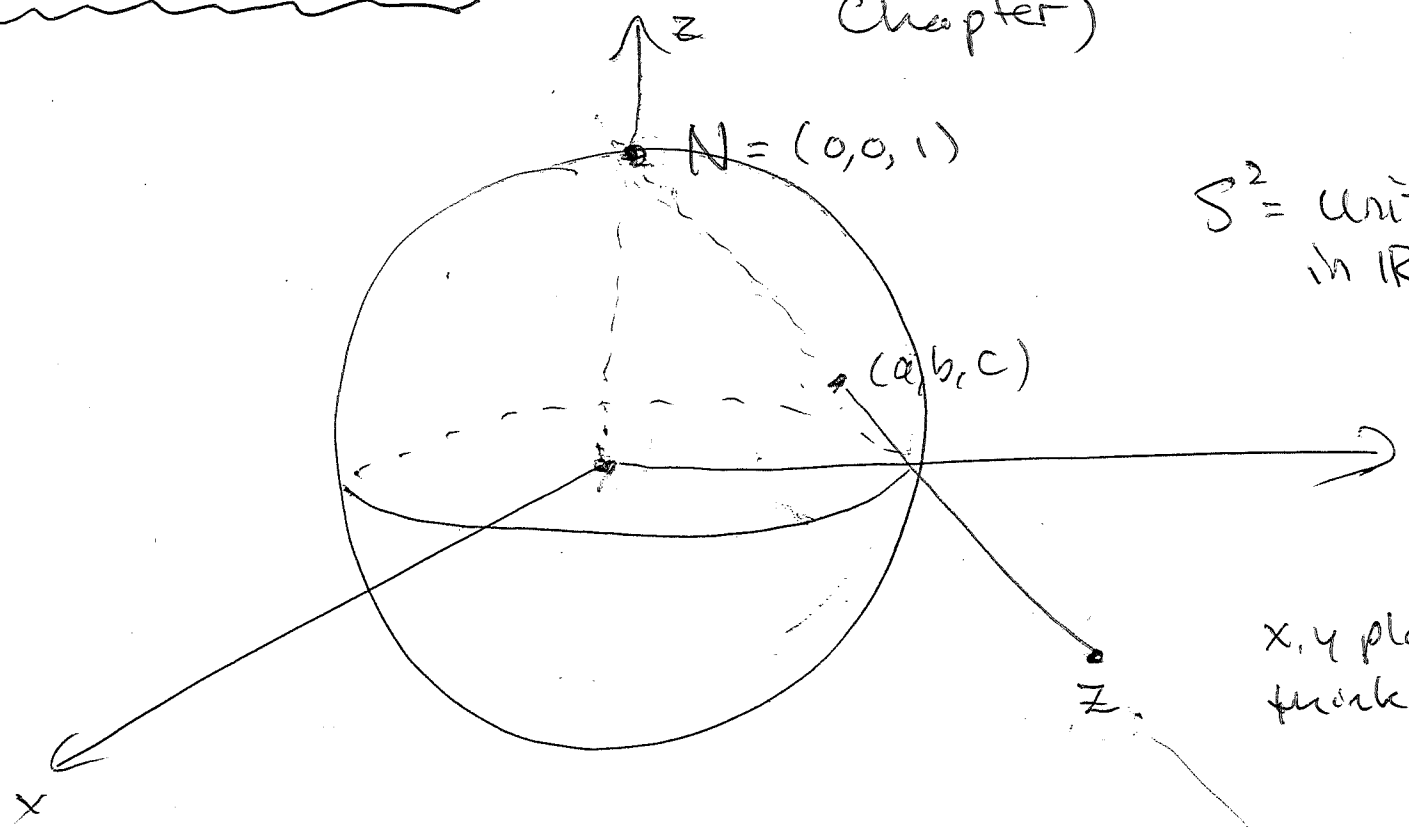
Thm: If $f(x+iy) = u(x,y) + iv(x,y)$ satisfies

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = 0$$

on some domain D, then

$$f = \text{const.} \quad \text{on } D$$

Riemann Sphere (1.7) (not needed for rest of Chapter)



S^2 = unit sphere in $\mathbb{R}^3$.

Consider the 'map'

$$\Psi : \mathbb{C} \longrightarrow S^2$$

$$z \longrightarrow (a,b,c)$$

explicitly,

$$\begin{cases} \Psi(x+iy) = \left(\dfrac{2x}{x^2+y^2+1}, \dfrac{2y}{x^2+y^2+1}, \dfrac{x^2+y^2-1}{x^2+y^2+1} \right) \\ \quad\quad = \left(\dfrac{2\,\mathrm{Re}(z)}{|z|^2+1}, \dfrac{2\,\mathrm{Im}(z)}{|z|^2+1}, \dfrac{|z|^2-1}{|z|^2+1} \right) \\ \Psi^{-1}(a,b,c) = \dfrac{a}{1-c} + i\,\dfrac{b}{1-c} \end{cases}$$

note: $\Psi(0) = (0,0,-1)$ = "S"

"$\Psi(\infty) = (0,0,1)$ = N"

Why bother with this map? and sphere?

note, any complex $f: \mathbb{C} \to \mathbb{C}$ can be viewed as map from $S^2 \to S^2$.

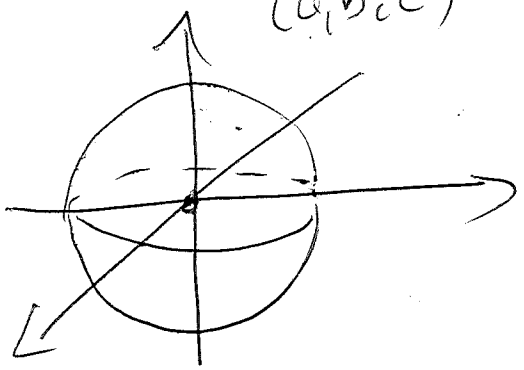
$$F = S^2 \to S^2$$

$$F = \Psi \circ f(\Psi^{-1}(a,b,c))$$

turns out that doing so is useful in describing singularities of certain analytic functions. Will illustrate below:

ex) $f(z) = -\frac{1}{z}$ defined on $\mathbb{C} \setminus \{0\}$.
there is singularity at $z=0$.

Come back to sphere later!



$a^2+b^2+c^2=1$

$\xrightarrow{\Psi^{-1}}$ $\frac{a}{1-c} + \frac{b}{1-c}\,i$

$\xrightarrow{-\frac{1}{z}}$ $\dfrac{-\left(\frac{a}{1-c} - i\,\frac{b}{1-c}\right)}{\frac{a^2+b^2}{(1-c)^2}(1-c^2)\,(1-c)(1+c)}$

$= \dfrac{-a}{1+c} + i\,\dfrac{b}{1+c}$

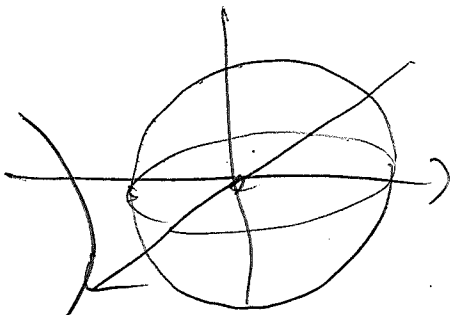
$\xrightarrow{\Psi}$ $\left(\dfrac{\frac{-2a}{1+c}}{\frac{a^2+b^2}{(1+c)^2}+1}, \cdots, \cdots\right)$

$\dfrac{(1-c^2)+(1+c)^2}{(1+c)^2}$

$= \dfrac{1-c^2+1+2c+c^2}{(1+c)^2}$ $\quad 2(1+c)$

$= \dfrac{2}{1+c}$

$= (-a, b, -c)$

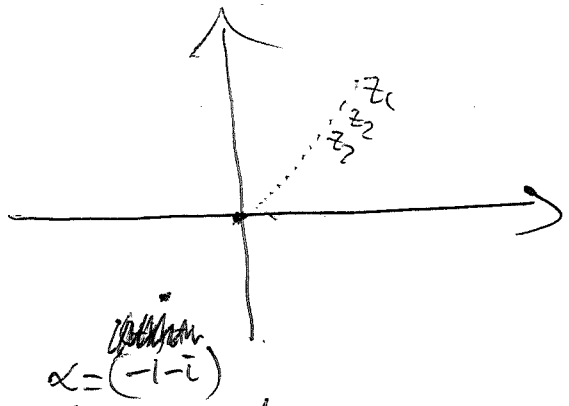


rotates S^2 by 180° about x^2 axis (y axis)

limits and continuity §2.2

Def^n $\lim_{n\to\infty} z_n = \alpha$ $\iff$ for any $\delta > 0$, $\exists\, N$ such that $|z_n - \alpha| < \delta$ for all $n > N$

eg) $z_n = \frac{1}{n} + \frac{i}{n}$



z_n gets 'closer & closer' to α as $n \to \infty$ but not "close enough". Thus we need more technical definition above.

eg) $z_n = \left(\frac{1+i}{5}\right)^n$

show using def^{n} that $\lim_{n\to\infty} z_n = 0$

$$z_n = \left(\sqrt{\tfrac{2}{25}}\, e^{i\pi/4}\right)^n$$

$$= \left(\frac{2}{25}\right)^{n/2} e^{i\pi n/4}$$

$$\therefore |z_n - 0| = \left(\frac{2}{25}\right)^{n/2}$$

now given any $\varepsilon > 0$, lets choose $N = \log_{2/25}(\varepsilon) \cdot 2$

then

$$n > N \iff n > 2\log_{2/25}(\varepsilon)$$

$$\iff |z_n - 0| = \left(\frac{2}{25}\right)^{n/2} \leq \left(\frac{2}{25}\right)^{2\log_{2/25}(\varepsilon)/2}$$

$$= \varepsilon$$

∎

eg) $\lim_{z_n \to 3i} \frac{z_n^2 + 9}{z_n - 3i} = \lim_{z_n \to 3i} \frac{(z_n + 3i)(z_n - 3i)}{(z_n - 3i)}$

$= 3i + 3i$

$= 6i$

usual tricks for limits apply.

eg) show

$$\lim_{n \to \infty} \left(\sum_{i=0}^{n} (i/2)^i \right) = \frac{1}{1 - i/2}$$

$$\lim_{n \to \infty} \left(1 + (i/2) + (i/2)^2 + \cdots (i/2)^n \right) = \frac{1}{1 - i/2}$$

(this is true when replace $i/2$ with real x in $(-1, 1)$)

$$= \lim_{n\to\infty} \left[\left(1 - \tfrac{1}{2}^2 + \tfrac{1}{2}^4 - \cdots\right) + \frac{i}{2}\left(1 - \tfrac{1}{2}^2 + \tfrac{1}{2}^4 - \cdots\right) \right]$$

$$= \lim_{n\to\infty} \left[\left(1 + \left(-\tfrac{1}{4}\right) + \left(-\tfrac{1}{4}\right)^2 + \cdots\right) + \frac{i}{2}\left(1 + \left(-\tfrac{1}{4}\right) + \left(-\tfrac{1}{4}\right)^2 + \cdots\right) \right]$$

we know sum of geom. series! for real numbers

$$= \left(\frac{1}{1-\left(-\frac{1}{4}\right)}\right) + \frac{i}{2}\left(\frac{1}{1-\left(-\frac{1}{4}\right)}\right)$$

$$= \left(\frac{4}{5}\right) + \frac{i}{2}\left(\frac{4}{5}\right) \sim$$

YES!

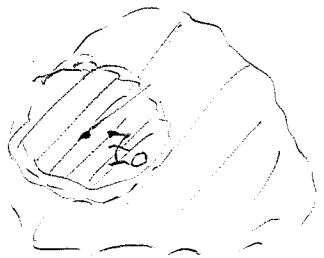
check! $\left(\frac{1}{1-i/2}\right) = \frac{1+i/2}{1+1/4} = \left(\frac{1}{1+1/4}\right) + \frac{i}{2}\left(\frac{1}{1+1/4}\right)$

$= 4/5 + i/2\ 4/5$!!!

limits of functions:

Def - let $f(z)$ be defined in nbrhd of z_0 (except possibly not at z_0)

$\lim_{z \to z_0} f(z) = w_0$ $\iff$ for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - w_0| \leq \varepsilon$ for all $0 < |z - z_0| \leq \delta$

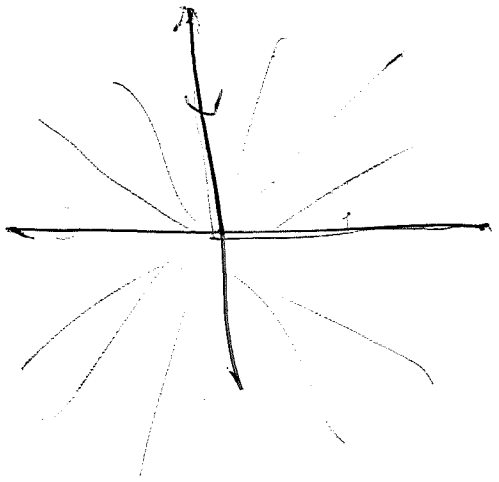


Def In particular, say $f(z)$ is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

* we say $\lim_{z \to z_0} f(z) = \infty$ $\Leftrightarrow$ given any $M > 0$, there exists $\delta > 0$ so that $|f(z)| \geq M$ for all $0 < |z - z_0| \leq \delta$

eg) Show $\lim_{z \to 0} \left(\frac{\bar{z}}{z} \right)$ does not exist!



$$= \lim_{re^{i\theta} \to 0} \left(\frac{re^{-i\theta}}{re^{i\theta}} \right)$$

$$= \lim_{r \to 0} e^{-i2\theta}$$

does not exist! any limit would depend also on θ!