Lect 3

eg) exponential form is convenient : $(re^{i\theta})$

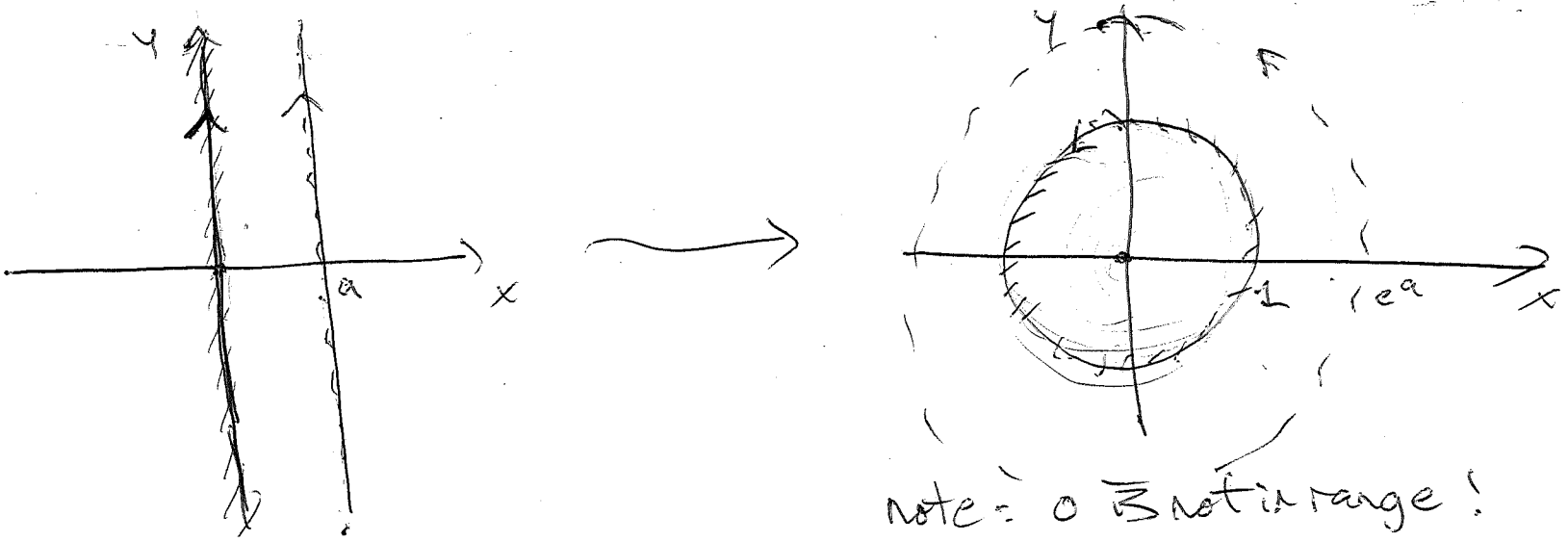
show $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

$$\overline{\left(\frac{1}{re^{i\theta}}\right)} = \overline{\left(\frac{1}{r}e^{-i\theta}\right)} = \frac{1}{r}\overline{e^{i\theta}} = \frac{1}{r}e^{i\theta}$$

$$\frac{1}{\bar{z}} \quad \frac{1}{\bar{z}} = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta}$$

eg) describe map $z \to e^z$ on plane $\left(x+iy \to e^x(\cos y + i\sin y)\right)$

$= e^x e^{iy}$



note: 0 ∉ not in range!

eg) Looking ahead: notice

$$\begin{cases} \mathrm{Re}(e^z) = e^x \cos y = u(x,y) \\ \mathrm{Im}(e^z) = e^x \sin y = v(x,y) \end{cases}$$

note that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$

likewise

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

z^β (complex powers) ; $(\beta \in \mathbb{Q})$

How do we define, say

$$z^{2/3}$$

for any complex z?

eg) $(1+i)^{2/3} = \alpha \iff (1+i)^2 = \alpha^3$

So we need to solve this for all complex α. Write as:

$\iff (\sqrt{2}e^{i\pi/4})^2 = (re^{i\theta})^3$

$\iff 2e^{i\pi/2} = r^3 e^{i3\theta}$

$\iff \begin{cases} r = \sqrt[3]{2} \\ 3\theta = \pi/2 + 2\pi n \end{cases} \quad n \in \mathbb{Z}$

$\iff \begin{cases} r = \sqrt[3]{2} \\ \theta = \pi/6 + \frac{2\pi n}{3} \end{cases} \quad n \in \mathbb{Z}$

$r = \sqrt[3]{2}$	$\theta =$
	$\pi/6 \pm \frac{2\pi}{3}\cdot 0$
	$\pi/6 \pm \frac{2\pi}{3}\cdot 1$
	$\pi/6 \pm \frac{2\pi}{3}\cdot 2$
	~~$\pi/6 \pm \frac{2\pi}{3}\cdot 3$~~ same
	~~$\pi/6 \pm \frac{2\pi}{3}\cdot 4$~~ same
	⋮

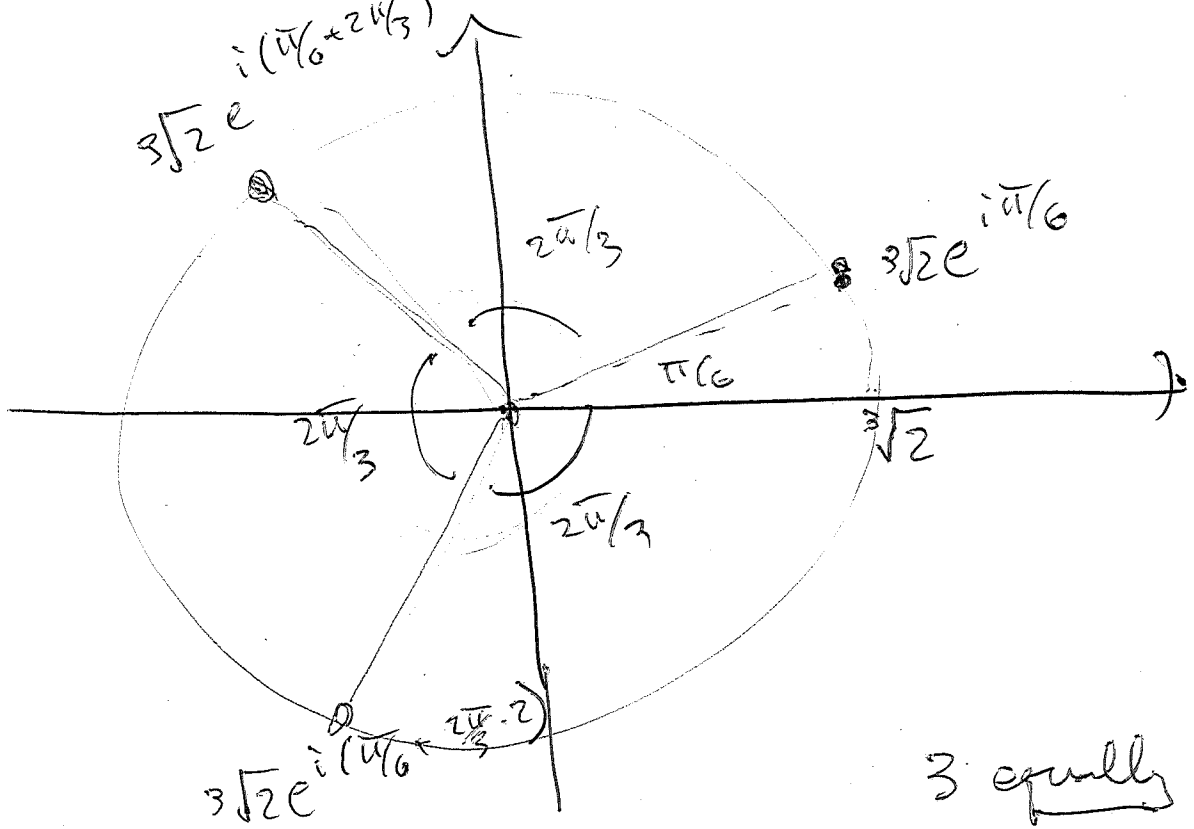
$i(\pi/6 + \frac{2\pi}{3}\cdot 1)$

$i(\pi/6 + \frac{2\pi}{3}\cdot 2)$

looks like ∞ many α's. but look!

the only distinct α's get are: $\sqrt[3]{2}e^{i\pi/6}$, $\sqrt[3]{2}e^{\quad}$, $\sqrt[3]{2}e^{\quad}$

plot these on plane!



3 equally spaced pts on circle, one is at $\sqrt[3]{2}\,e^{i\pi/6}$.

Def ^{n} given pos. integrs $m, n \in \mathbb{Z}^+$

(1) $(re^{i\theta})^{1/n} := \sqrt[n]{r}\, e^{i(\theta/n + 2\pi K/n)}$

~~$K = 0, \pm 1, \pm 2, \ldots$~~

$K = 0, 1, 2, \ldots, (n-1)$

(2) $(re^{i\theta})^{m/n} := \left[(re^{i\theta})^m\right]^{1/n}$

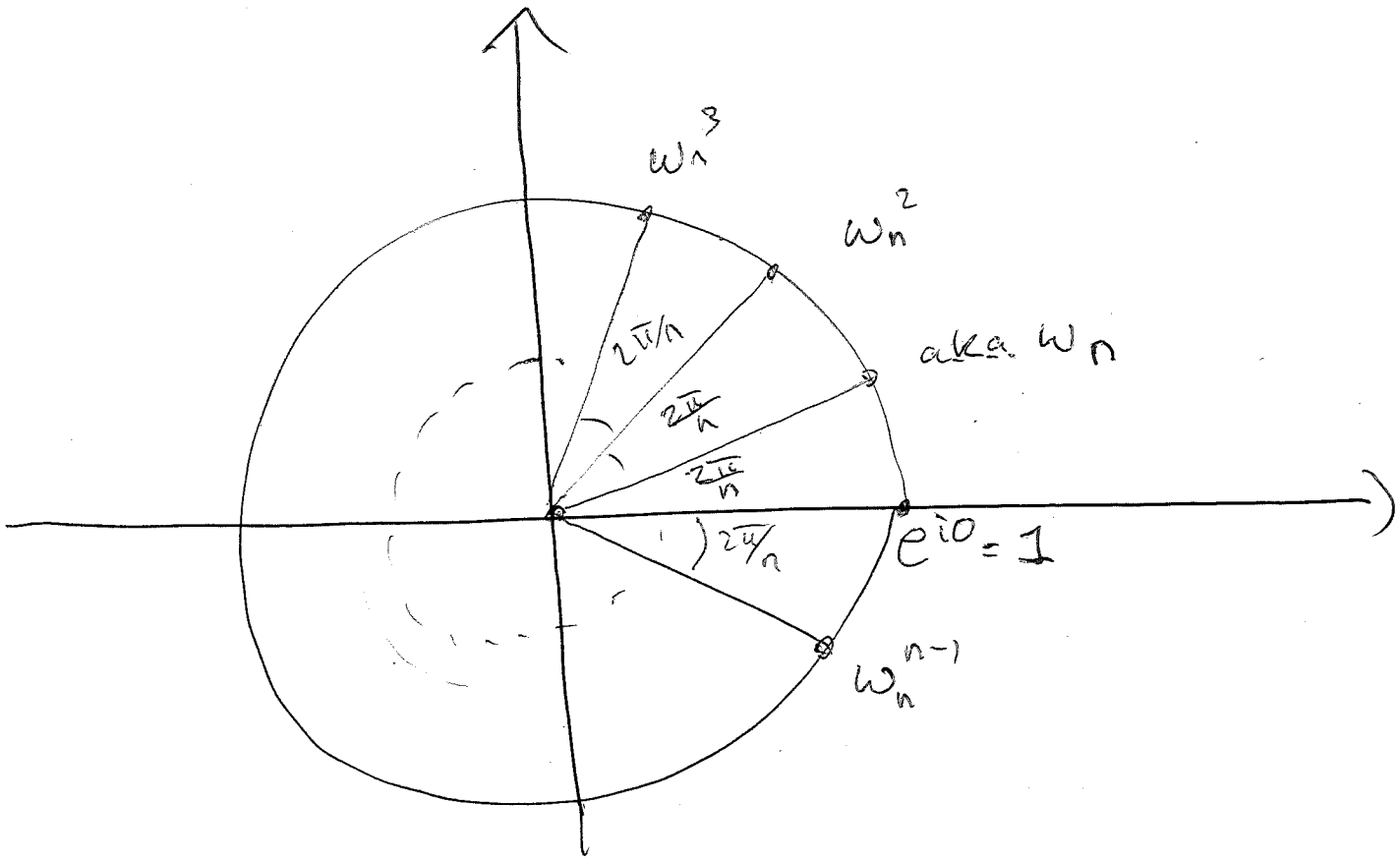
(equivalently, could write (1) as $z^{1/n} = |z|^{1/n} e^{i\frac{\arg(z)}{n}}$)

eg) the n-th roots of unity are the n solutions to

$$\boxed{z^n = 1}$$

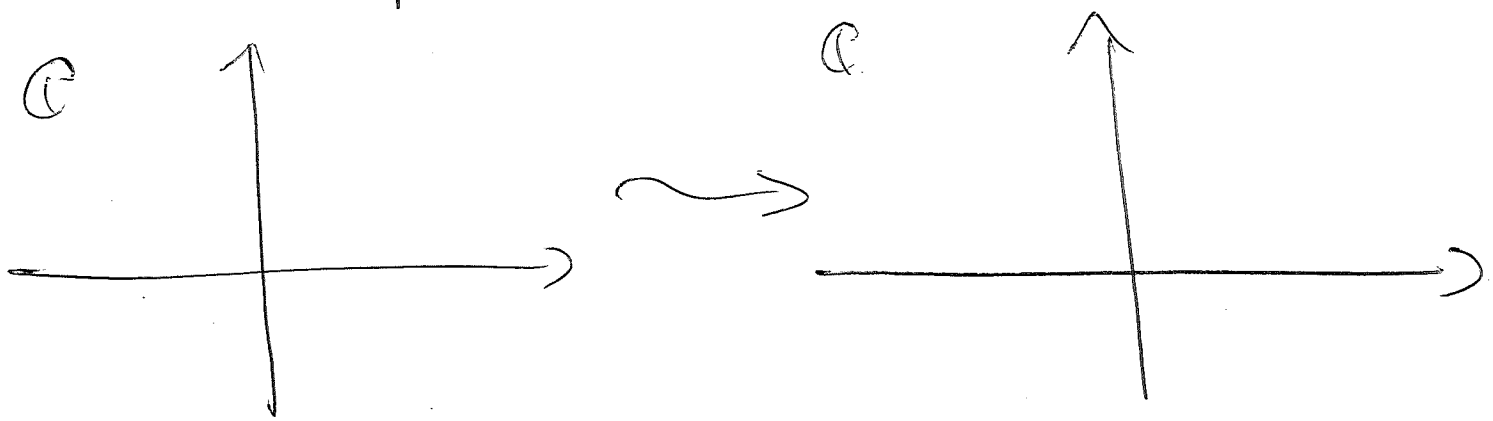
$$\Leftrightarrow z = \sqrt[n]{1} = \left\{1 = e^{i0},\ e^{i(0+\frac{2\pi}{n})},\ e^{i(0+\frac{2\pi}{n}\cdot 2)},\ \ldots,\ e^{i(0+\frac{2\pi}{n}(n-1))}\right\}$$

$$= \left\{e^{i0},\ \underbrace{e^{i\frac{2\pi}{n}}}_{\omega_n},\ \underbrace{e^{i\frac{2\pi}{n}\cdot 2}}_{\omega_n^2},\ \ldots,\ \underbrace{e^{i\frac{2\pi}{n}(n-1)}}_{\omega_n^{n-1}}\right\}$$



the n roots of unity !

eg) how does map $z \to z^{1/2}$ act on plane?



not well defined! $f(z) = z^{1/2}$ is multiple (double) valued!

namely

$$z \to z^{1/2} = |z|^{1/2} e^{i \frac{\arg(z)}{2}}$$

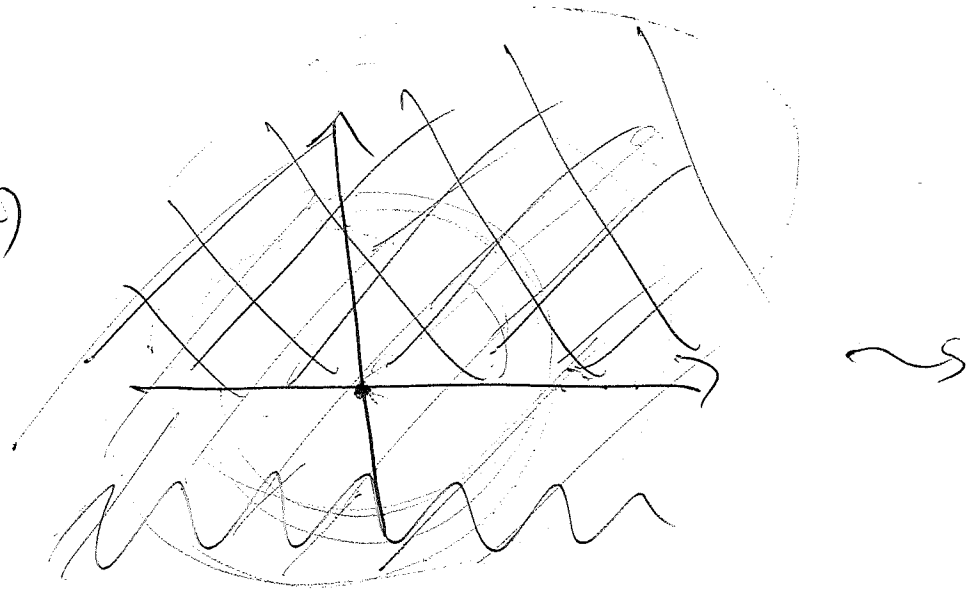
is doubly defined. However, it is well defined if we choose a branch of $\arg z$.

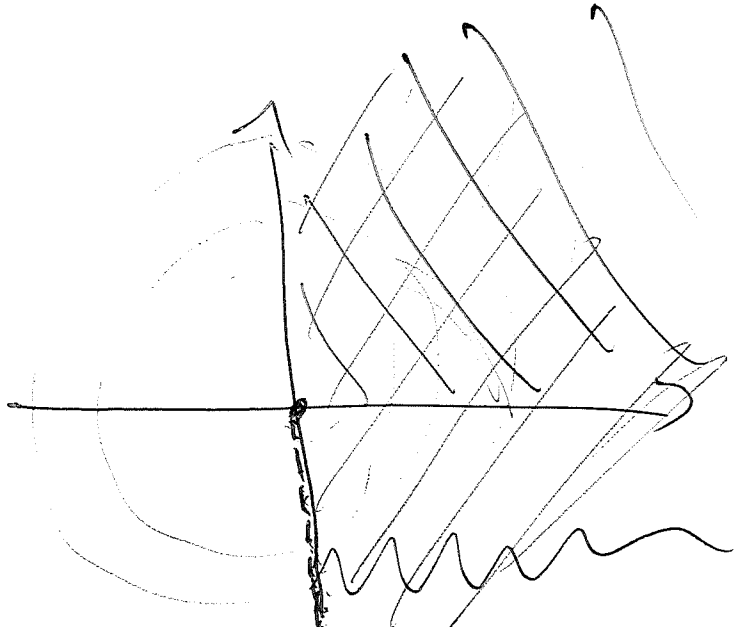
eg) describe map $z \to |z|^{1/2} e^{i\,\mathrm{Arg}(z)/2}$ on plane

a)

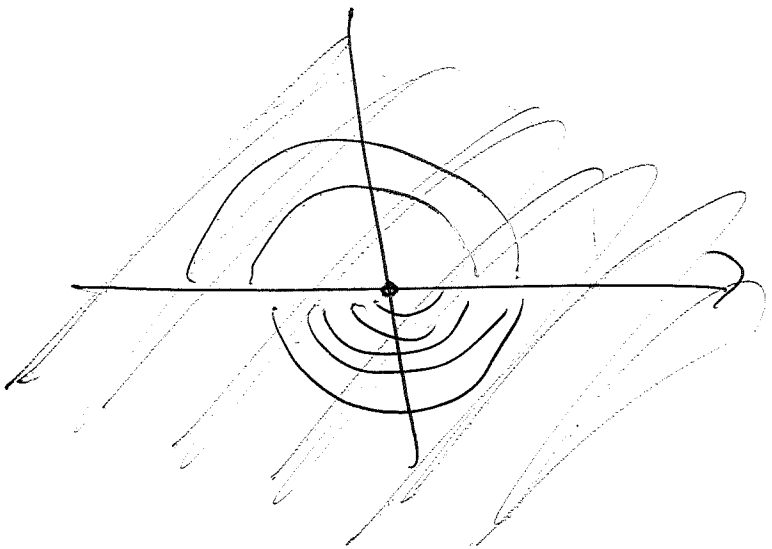
b) " " $z \to |z|^{1/2} e^{i\arg(z)/2}$ on plane

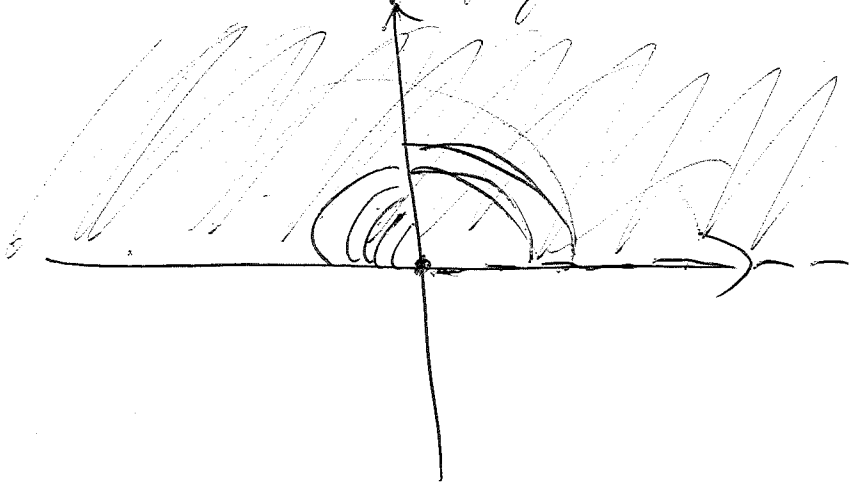
a)





b)





(§2.1) Functions of a complex variable

A complex function:

$$\begin{cases} f : \mathbb{C} \longrightarrow \mathbb{C} \\ (x+iy) \longrightarrow u(x,y) + iv(x,y) \end{cases}$$

Some
$u: \mathbb{R}^2 \to \mathbb{R}$
$v: \mathbb{R}^2 \to \mathbb{R}$

as see before, may write any such f as depending on x, y as above, or as depending on $z, \bar{z}$:

$$\left\{ \begin{array}{l} z = x+iy; \ \bar{z} = x-iy \\ x = \frac{z+\bar{z}}{2} \ ; \ y = \frac{z-\bar{z}}{2i} \end{array} \right\}$$

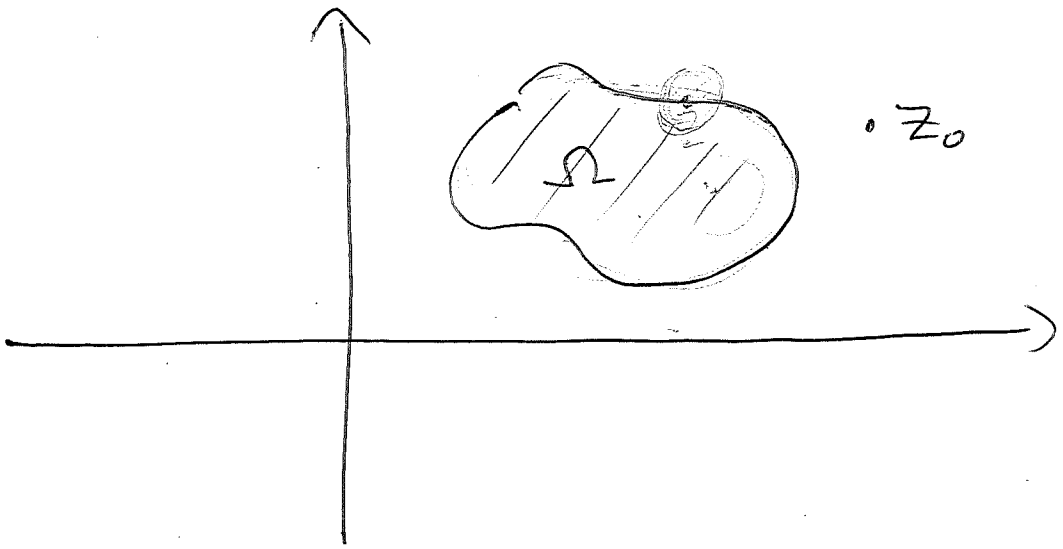
eg) $f(x+iy) = x^2 + iy$

$$= \left(\frac{z+\bar{z}}{2}\right)^2 + i\left(\frac{z-\bar{z}}{2i}\right)$$

$$= \frac{z^2}{4} + \frac{z\bar{z}}{2} + \frac{\bar{z}^2}{4} + \frac{z}{2} - \frac{\bar{z}}{2} \quad \cdots$$

insert here, standard discussion on "domain, Range" of a complex function ...

("topology" of $\mathbb{C}$):



* $D_\delta(z_0)$ = open disc radius δ around z_0 $\{z \mid |z-z_0| < \delta\}$

* Ω is open : $z \in \Omega \Rightarrow D_\delta(z) \subset \Omega$

* Ω is closed : Ω^c, its compliment is open

* Ω is connected : every pair $z_1, z_2 \in \Omega$ can be joined by piecewise linear path inside Ω.