

--- from Quiz #0 ---

$$\frac{z}{1-z} = (1-s_i) = \alpha$$

$$\Leftrightarrow z = \alpha(1-z)$$

$$\Leftrightarrow z = \alpha - \alpha z$$

$$\Leftrightarrow z(1+\alpha) = \alpha$$

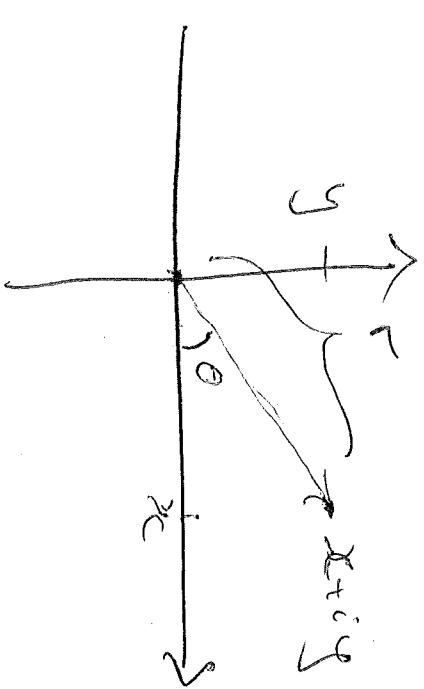
$$\Leftrightarrow z = \frac{\alpha}{1+\alpha} = \frac{1-s_i}{2-s_i} \cdot \frac{2+s_i}{2+s_i} = \frac{(2+2s_i) + i(-10+s_i)}{2^2+s_i^2} = \frac{27}{29} - \frac{s_i}{29}$$

Polar form { Consequences }

Given a complex number

$$\boxed{z = x+iy}$$

(rectangular form)



Lecture 2

①

Can also write

(2)

$$\boxed{z = r(\cos \theta + i \sin \theta)}$$

(polar form)

$$\left\{ \begin{array}{l} r = |z| \\ \theta = \tan^{-1}(y/x) \end{array} \right. \text{as long as } x \neq 0$$



Following computation unlocks all mystery --etc-- to complex multiplication!

$$\begin{aligned} z_1 z_2 &= r_1 (\cos \theta_1 + i \sin \theta_1) r_2 (\cos \theta_2 + i \sin \theta_2) \\ (3) \quad &= r_1 r_2 \left[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2) \right] \\ &= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right] \end{aligned}$$

TRIG IDENTITIES

- "multiply the moduli"
- "add the corresponding angles!"

(3)

(3)

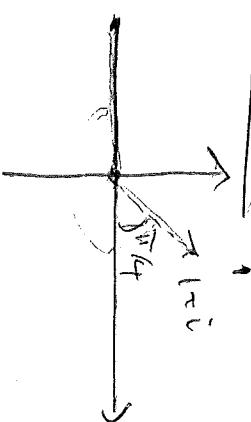
This is the key to § 1.3, 1.4, 1.5. Before this, must deal with an annoying fact: "angle of z is not well defined". We deal with this by introducing a formal:

Def'n: given $z \neq 0$, we define

such that

$$\arg(z) := \text{set of all } \theta \in \mathbb{R} \ni z = |z|(\cos \theta + i \sin \theta)$$

$$\begin{aligned}\arg_z(z) &:= \text{the unique } \theta \in \mathbb{R} \ni z = |z|(\cos \theta + i \sin \theta) \\ &\theta \in [\tau, 2\pi + \tau] \\ &(\text{branch of } \arg(z))\end{aligned}$$



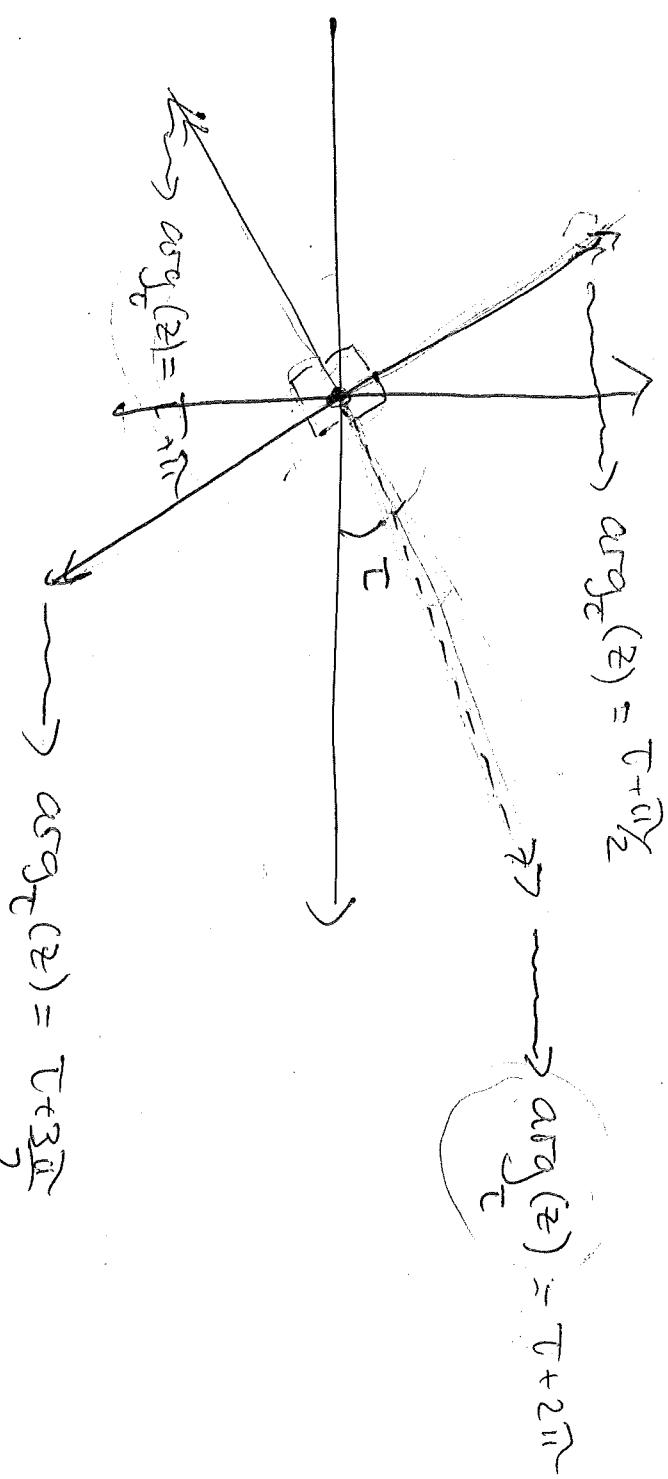
* $\arg(1+i) = \left\{ \frac{\pi}{4} + 2\pi n \mid n \in \mathbb{Z} \right\}$

$\arg_n(1+i) = \frac{\pi}{4} + 2\pi n$

* $\arg(z)$ is thought as a multi-valued real function on set $\mathbb{C} \setminus \{0\}$.

* $\arg_{-\pi}(z) =: \operatorname{Arg}(z)$ is called principal branch of $\arg(z)$.

(4)



(dotted line called "branch cut" for $\arg_T(z)$).

In summary:

- $z_1 + z_2$: result of adding z_1, z_2 "as vectors"
- $|z_1 \cdot z_2| = |z_1||z_2|$
- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$
- $\arg(z_1 + z_2) = \arg(z_1) + \arg(z_2)$
" $S_1 = S_2 + S_3 \Leftrightarrow$ every elt. of S_1 is sum of elts. of S_2, S_3 . And vice versa.

ex) describe how following maps "act on plane \mathbb{C} "

a) $z \rightarrow (1+i)z$

b) $z \rightarrow z^2$

c) $z \rightarrow (\bar{z})^2$

a)

\mathbb{C}

The map rotates
 \mathbb{C} by $\frac{\pi}{4}$ clockwise (counter)
and dilates it by $\sqrt{2}$!

$$\sqrt{2}z$$

\mathbb{C}

b)

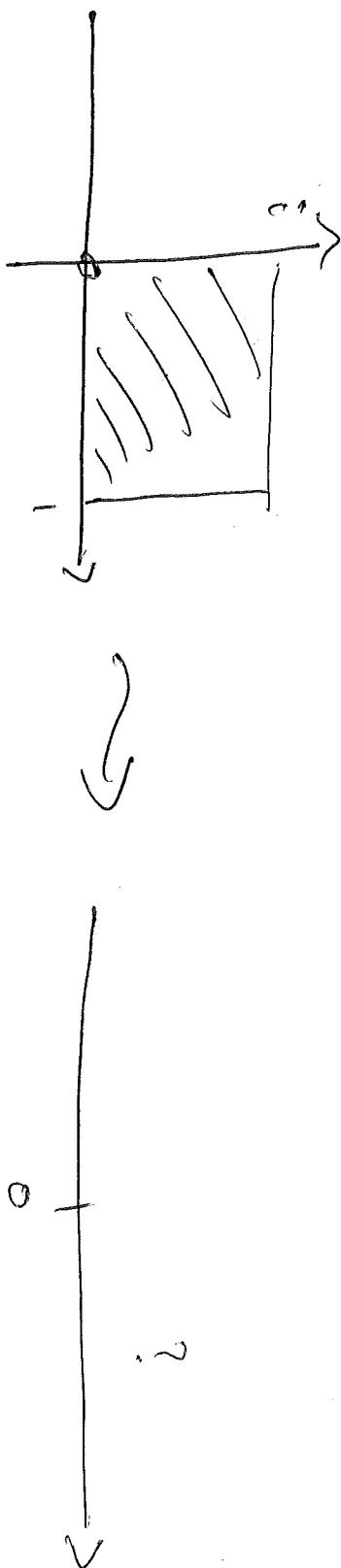


$$r(\cos\theta + i\sin\theta) \rightarrow r^2(\cos 2\theta + i\sin 2\theta)$$

the map "wraps \mathbb{C} around itself twice"

Q) image of shaded region under map $z \mapsto \arg_{-\pi}^{\pi}(z)$?

(6)



exponential & powers:

Calculation (3) gives access to the functions

$$z^\alpha, \quad z^\beta$$

$$\begin{cases} \alpha > 0 \text{ real} \\ \beta \text{ rational} \end{cases}$$

(exponential & power functions).

e^z (complex exponential):

Suffices to describe e^z , for $z = e^{(\ln x)}z$.

Observe what happens if replace r_1, r_2 in (3) with e^{r_1}, e^{r_2} :

$$e^{r_1} (\cos \theta_1 + i \sin \theta_1) e^{r_2} (\cos \theta_2 + i \sin \theta_2)$$

$$= e^{r_1+r_2} (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)$$

$$= e^{r_1+r_2} (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

and so, if we let $f(x+iy) = e^x (\cos y + i \sin y)$, get

$$f(z_1) \cdot f(z_2) = f(z_1 + z_2)$$

Notations:

$$\text{Def: } e^{x+iy} := e^x (\cos y + i \sin y)$$

* and so, we see:

$$e^z e^{z_2} = e^{z_1 + z_2}$$

also, "extends e^x " from real line!

$$\begin{aligned} e^x &= e^{x+i0} \\ &= e^x (\cos 0 + i \sin 0) \\ &= e^x (1 + i0) \\ &= e^x \end{aligned}$$

* Def'n is consistent with Power series def'n of exp.

Instead, define $\rightarrow e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$

$$\begin{aligned} &= 1 + (x+iy) + \frac{(x+iy)^2}{2!} + \frac{(x+iy)^3}{3!} + \frac{(x+iy)^4}{4!} + \dots \\ &= \underbrace{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)}_{=} \underbrace{\left(1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots\right)}_{=} \end{aligned}$$

$$\begin{aligned} &= e^x \left[\left[-\frac{y^2}{2!} + \frac{y^4}{4!} - \dots \right] + i \left[y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right] \right] \\ &= e^x (\cos y + i \sin y) \end{aligned}$$

X

(9)

This is alternate approach to e^z .
This way emphasizes pure dependence of e^z on z alone!
(not on \bar{z}).

* Note the identities :

$$\left. \begin{aligned} \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{aligned} \right\}$$

In particular, we'll use these later to define $\cos z$, $\sin z$
for complex z !!

We may write polar form as:

$$\boxed{z = r e^{i\theta}}$$

$$\left\{ \begin{array}{l} r = |z| \\ \theta \in \arg(z) \end{array} \right.$$

* $e^{i\theta}$ describes a complex # on unit circle.

$$\left\{ \begin{array}{l} 1 = e^{i2\pi n} \quad n \in \mathbb{Z} \\ -1 = e^{i(\pi + 2\pi n)} \\ i = e^{i(\pi/2 + 2\pi n)} \\ -i = e^{-i(\pi/2 + 2\pi n)} \end{array} \right.$$

(10)