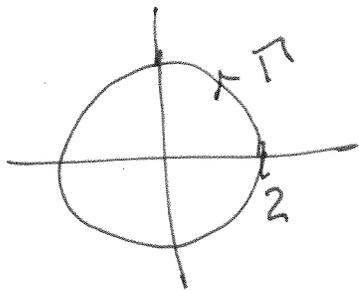


# SOLUTIONS :

§ 4.5 3c)  $\int_{\Gamma} \frac{\cos z}{z(z^2+9)} dz = \int_{\Gamma} \left( \frac{\cos z}{(z+3i)(z-3i)} \right) \frac{dz}{z}$

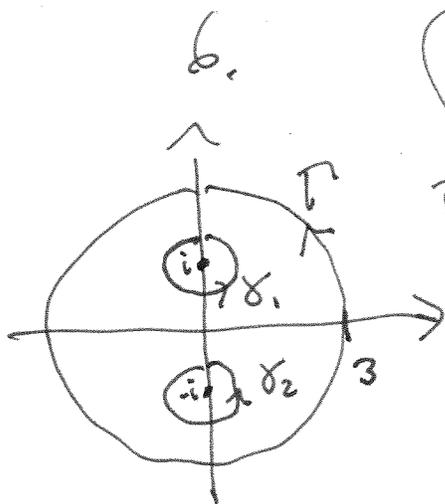


$$= 2\pi i \left( \frac{\cos z}{z^2+9} \right) \Big|_{z=0}$$

$$= 2\pi i \left( \frac{1}{9} \right)$$

d)  $\int_{\Gamma} \frac{5z^2+2z+1}{(z-i)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} (5z^2+2z+1) \Big|_{z=i}$

$$= \pi i (10)$$



$$\int_{\Gamma} \frac{e^{iz}}{(z^2+1)^2} dz = \int_{\Gamma} \frac{e^{iz}}{(z+i)^2(z-i)^2} dz = \int_{\gamma_1} + \int_{\gamma_2}$$

(by Cauchy or deformation thm as § 4.4)

$$= \frac{2\pi i}{1!} \frac{d}{dz} \left( \frac{e^{iz}}{(z+i)^2} \right) \Big|_{z=i} + \frac{2\pi i}{1!} \frac{d}{dz} \left( \frac{e^{iz}}{(z-i)^2} \right) \Big|_{z=-i}$$

8. Done in Lecture 19.

10. 
$$\int \frac{f'(z)}{z-z_0} dz = 2\pi i f'(z_0) \quad \text{Cauchy Formula (Theorem 14)}$$

$$= 2\pi i \left( \frac{1}{2\pi i} \int \frac{f(z)}{(z-z_0)^2} dz \right) \quad \text{Cauchy Formula for derivatives (Theorem 19)}$$

11.  $f = u + iv$  analytic  $\Rightarrow$  all partial der. of  $u, v$  exist (since  $f', f'', f''', \dots$  exist).

and also a harmonic

(i.e.)  $u_{xx} + u_{yy} = 0$

(by § 2.5)

$$\Rightarrow (u_{xx})_{xx} + (u_{yy})_{xx} = 0$$

$$\Rightarrow (u_{xx})_{xx} + (u_{xx})_{yy} = 0$$

(equality of mixed partial der.)

$$\Rightarrow \underline{u_{xx} \text{ harmonic}}$$

$$\S 4.6 \# 4. \quad |a_k| = \left| \frac{1}{k!} \frac{d^k}{dz^k} P(z) \right|_{z=0}$$

$$= \frac{1}{k!} \left| \frac{k!}{2\pi i} \int_{|z|=1} \frac{P(\omega)}{(\omega-0)^{k+1}} d\omega \right|$$

$$\leq \frac{1}{2\pi} \cdot \frac{M}{1^{k+1}} \cdot (2\pi)$$

$$= M$$

#6.  $f^{(5)}$  bounded + entire

$$\Rightarrow f^{(5)}(z) = \alpha_0 \quad \text{some } \alpha_0$$

$$\Rightarrow f^{(4)}(z) = \alpha_0 z + \alpha_1 \quad \text{some } \alpha_1$$

$$\Rightarrow f^{(3)}(z) = \frac{\alpha_0 z^2}{2} + \alpha_1 z + \alpha_2 \quad \text{some } \alpha_2$$

⋮

$$\Rightarrow f(z) = \frac{\alpha_0 z^5}{5!} + \frac{\alpha_1 z^4}{4!} + \dots + \alpha_5$$

# 9.  $\left\{ \begin{array}{l} \text{suppose } |f(w)| \leq |f(z_0)| \text{ on } C_R \\ \text{and } |f(z_0 + Re^{it})| < |f(z_0)| - \varepsilon; \quad 0 \leq a \leq t \leq b \leq 2\pi \end{array} \right.$

$$\Rightarrow |f(z_0)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + Re^{it}) dt \right|$$

$$\leq \frac{1}{2\pi} \left[ \left| \int_0^a f(z_0 + Re^{it}) dt \right| + \left| \int_a^b f(z_0 + Re^{it}) dt \right| + \left| \int_b^{2\pi} f(z_0 + Re^{it}) dt \right| \right]$$

$$\leq \frac{1}{2\pi} \left[ |f(z_0)| \cdot a + (|f(z_0)| - \varepsilon)(b-a) + |f(z_0)| \cdot (2\pi - b) \right]$$

$$= |f(z_0)| - \frac{\varepsilon(b-a)}{2\pi} |f(z_0)|$$

contradiction

# 15.  $f$  has no zero in  $D$

$\Rightarrow \frac{1}{f}$  analytic in  $D$

$\Rightarrow \frac{1}{|f(z)|} \leq \frac{1}{M}$  all  $z$  in  $D$

$\Rightarrow |f(z)| \geq M$  all  $z$  in  $D$

$\Rightarrow f$  constant in  $D$

$\Rightarrow$  contradiction

(Maximum Mod. principle)

(Max. Mod. Principle  
where  $M = \text{value of } |f| \text{ on } B$ )

(Maximum Mod. principle)

$$\# 18. \text{ (by } \S 3.1 \text{ \#17): } p(z) = a_n(z-z_1)^{d_1}(z-z_2)^{d_2}\dots(z-z_r)^{d_r}$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\Gamma} \frac{p'}{p} = \frac{1}{2\pi i} \int_{\Gamma} \frac{d_1}{z-z_1} + \frac{d_2}{z-z_2} + \dots$$

$$= \frac{d_1}{2\pi i} \int_{\Gamma} \frac{1}{z-z_1} + \frac{d_2}{2\pi i} \int_{\Gamma} \frac{1}{z-z_2} + \dots$$

$$= \frac{d_1 \cancel{2\pi i}}{2\pi i} + \frac{d_2 \cancel{2\pi i}}{2\pi i} + \dots$$

$$= d_1 + d_2 + \dots + d_r$$