

HW # 4

§2.4 #2 $\begin{cases} u_x = 3x^2 + 3y^2 - 3 = v_y \\ u_y = 6xy ; -v_x = -6xy \end{cases}$

\Rightarrow CR equations hold iff $6xy = -6xy$
 iff $xy = 0$
 iff $x=0$ or $y=0$

$\Rightarrow f$ differentiable only on axis x or y ,
 thus f not diff on any open set
 thus f nowhere analytic.

#3. $\begin{cases} u_x = 6x+2 = v_y \\ u_y = -6y = -v_x \end{cases}$

\Rightarrow CR equations satisfied everywhere
 $\Rightarrow f$ diff everywhere, thus analytic everywhere.

#6. $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$

$$= \lim_{r \rightarrow r_0} \frac{(u(r, \theta_0) + i v(r, \theta_0)) - (u(r_0, \theta_0) + i v(r_0, \theta_0))}{re^{i\theta_0} - r_0 e^{i\theta_0}} = (r - r_0) e^{i\theta_0}$$

$$= \underbrace{u_r(r_0, \theta_0) + i v_r(r_0, \theta_0)}_{e^{i\theta_0}}$$

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\theta \rightarrow \theta_0} \frac{[U(r_0, \theta) + iV(r_0, \theta_0)] - [U(r_0, \theta_0) + iV(r_0, \theta_0)]}{r_0 e^{i\theta} - r_0 e^{i\theta_0}}$$

$$= \lim_{\theta \rightarrow \theta_0} \frac{[U(r_0, \theta) - U(r_0, \theta_0)] - [V(r_0, \theta) - V(r_0, \theta_0)]}{(\theta - \theta_0)} \cdot \frac{1}{ir_0} \cdot \frac{1}{\frac{e^{i\theta} - e^{i\theta_0}}{(i\theta - i\theta_0)}}$$

$$= (U_\theta + iV_\theta) \cdot \frac{1}{ir_0} \cdot \frac{1}{e^{i\theta_0}}$$

$$= \frac{V_\theta/r_0 - iU_\theta/r_0}{e^{i\theta_0}}$$

$U_r = U_\theta/r$
$V_r = -U_\theta/r$

#13. $u(r, \theta) = r^n \sin(n\theta)$ is harmonic in wedge
 and $= \begin{cases} 0 & \text{on non-negative } x\text{-axis} \\ r^n \sin(n\pi/4) & \text{on the } y=x \end{cases}$
 thus $u(r, \theta) = r^4 \sin 4\theta$ is harmonic in wedge and $\neq 0$ along both edges

13. $f = u + iv$ and $|f| = \sqrt{u^2+v^2}$, hence $|f|^2 = u^2+v^2$, are all analytic on D :

$$\Rightarrow \begin{cases} (u^2+v^2)_x = 2uu_x+2vv_x = 0 \\ (u^2+v^2)_y = 2uu_y+2vv_y = 0 \end{cases} \quad (\text{CR eqn for } |f|^2)$$

$$\Rightarrow |f|^2 = u^2+v^2 = \text{const. in } D$$

$$\Rightarrow \begin{cases} 2uu_x+2vv_x = 0 \\ 2u(-v_x)+2v(u_x) = 0 \end{cases} \quad (\text{CR eqn for } f)$$

$$\Rightarrow 2u^2u_x+2v^2u_x = 0$$

$$\Rightarrow 2(u^2+v^2) \cdot u_x = 0$$

$$\Rightarrow u^2+v^2=0 \quad (\because |f|^2=0 \text{ in } D \quad \boxed{\therefore f=0 \text{ in } D})$$

OR

$$u_x=0 \text{ in } D \quad \left| \begin{array}{l} (\because f = \underline{\text{const in } D}) \\ (\text{and can likewise show } v_y=0 \text{ in } D) \end{array} \right.$$

Note: Once we know $|f|^2 = \text{const}$, may also argue $\Rightarrow f'(z) = 0$ on $D \rightarrow$

$\Rightarrow f'(z_0) = 0$ since $f'(z_0) \neq 0$
 $\Rightarrow f(D)$ contains some open disk (by Linear Model)
 $\therefore f(D)$ is not const

$\Rightarrow u_x+iV_x=0$ on D
 $V_y-iU_y=0$
 $\Rightarrow f = \text{const. on } D$.

#14. f maps domain D to portion of line $y=mx+b$
 $(m, b \text{ real})$

$$\Rightarrow f = u + iV \text{ and } V = mu + b \quad ①$$

$$\Rightarrow \begin{cases} u_x = V_y = m u_y \\ \text{CR} \end{cases} \quad ①$$

$$\begin{cases} u_y = -V_x = -mu_x \\ \text{CR} \end{cases} \quad ②$$

$$\Rightarrow u_x = -m^2 u_x \quad \therefore \begin{cases} (1+m^2)u_x = 0 \\ (1+m^2)u_y = 0 \end{cases}$$

$$u_y = -m^2 u_y$$

$$\Rightarrow u_x = 0 \quad \text{in } D \quad \therefore V_x = 0 \quad \text{in } D$$

$$u_y = 0 \quad \text{CR} \quad V_y = 0$$

$$\Rightarrow f = \text{const. in } D$$

May also correctly observe that $\tilde{f}(z) = \alpha f(z) + \beta$
maps D to real axis for appropriate α, β .

$$\Rightarrow \tilde{f} = \tilde{u} + i\tilde{V} = \tilde{u} + i0 \Rightarrow \begin{cases} \tilde{u}_x = 0 \\ \tilde{u}_y = 0 \end{cases} \quad \text{CR}$$

$$\Rightarrow \tilde{u} = \text{const.}$$

$$\Rightarrow \tilde{f} = \text{const.}$$

$$\Rightarrow f = \text{const.}$$

$$\S 2.5 \quad 3c) \quad u(x,y) = \frac{1}{2} \ln(x^2+y^2)$$

$$\text{Let } V(x,y) = - \int u_y(x,y) dx + f(y)$$

$$= - \int \frac{y}{x^2+y^2} dx + f(y)$$

$$= -\arctan\left(\frac{x}{y}\right) + f(y)$$

then $V_x = -u_y$

$$\text{Also, check } V_y = \frac{-1}{(x^2+y^2)^2} \left(-\frac{x}{y^2}\right) + f'(y)$$

$$= \frac{x}{y^2+x^2} + f'(y)$$

$$= u_x + f'(y)$$

Then $V_y = u_x$ if $f' = \text{const.}$

so $V = -\arctan\left(\frac{x}{y}\right)$ is harmonic conjugate in $\operatorname{Re}(z) > 0$

(Note: $V = -(\pi/2 - \arctan(y/x))$)

$$= \arctan\left(\frac{y}{x}\right) - \pi/2 \quad \text{as long as } x, y \neq 0$$

#13. $u(r, \theta) = r^n \sin(n\theta)$ harmonic in wedge
 and = $\begin{cases} 0 & \text{on non-neg } x \text{ axis} \\ r^n \sin\left(\frac{n\pi}{4}\right) & \text{on line } y=x \end{cases}$

thus $u=r^4 \sin(4\theta)$ is function we want

$$\begin{aligned} \# 18. \quad \phi \text{ haroni} &\Rightarrow \phi_{xx} + \phi_{yy} = 0 \\ &\Rightarrow (\phi_x)_x = (-\phi_y)_y \\ &\Rightarrow \begin{cases} (\phi_x)_x = (-\phi_y)_y \\ (\phi_x)_y = -(-\phi_y)_x \end{cases} \quad \begin{matrix} \text{just } \phi_{xy} = \phi_{yx} \\ \text{by given} \\ \text{assumption} \end{matrix} \\ &\Rightarrow \phi_x + (-\phi_y)i \\ &\text{analytic} \end{aligned}$$

21. 3e) (§2.4) shows that any haroni conjugate of u looks like

$$\begin{aligned} v &= \arctan(y/x) + C \quad (\text{if } x, y \neq 0) \\ &= \underline{\text{Arg}(z) + C} \quad \text{which is not even} \\ &\quad \text{continuous on neg} \\ &\quad \text{real axis (neg } x \text{ axis)} \end{aligned}$$