

$$[5] \quad 1. \quad a) \quad \frac{(5+i)}{1+5i} = \frac{(5+i)(1-5i)}{1^2 + 5^2} = \boxed{\frac{10-24i}{26}}$$

$$b) \quad (\sqrt{3}+i)^{18}(2+2i) = (2e^{i\pi/6})^{18} (2\sqrt{2}e^{i\pi/4})$$

$$= 2^{18} e^{i\pi/3} \cdot 2\sqrt{2} e^{i\pi/4}$$

$$= \boxed{2^{19} \sqrt{2} e^{i(13\pi/4)}}$$

2. Find all solutions to following, express as a+bi

$$a) \quad z^4 + z^3 + z^2 = 0 \quad \Rightarrow \quad z^2(z^2 + z + 1) = 0$$

$$\Rightarrow \quad z=0, \quad -\frac{1}{2} \pm \frac{\sqrt{1-4}}{2}$$

$$= \boxed{0, \quad -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i}$$

$$b) \quad \log(z-2) = 1 + i\frac{\pi}{2}$$

$$\Rightarrow \ln|z-2| + i\arg(z-2) = 1 + i\frac{\pi}{2}$$

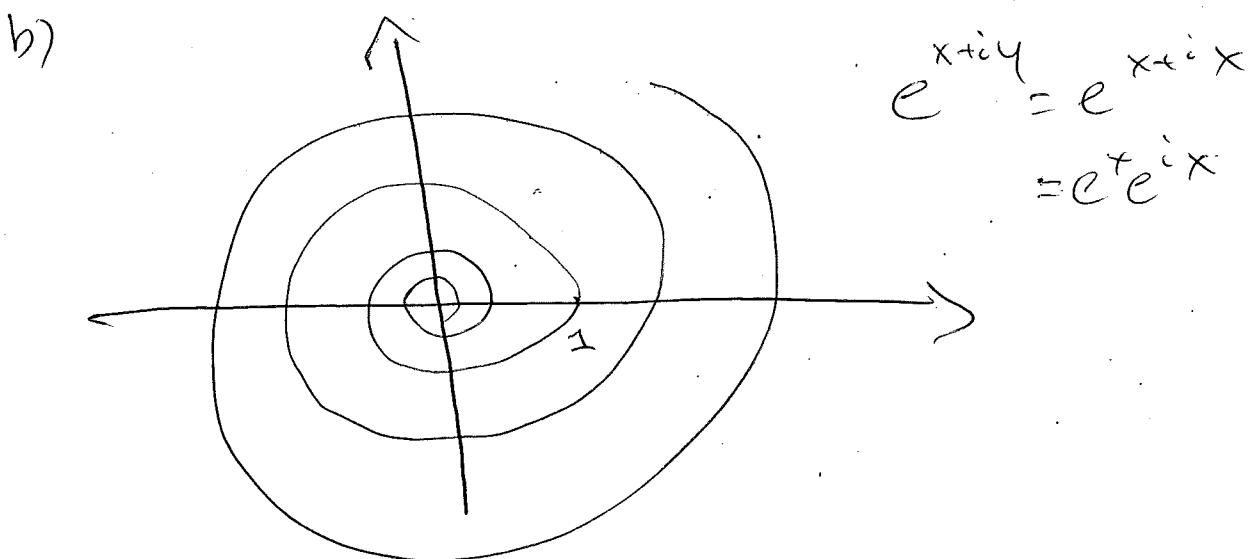
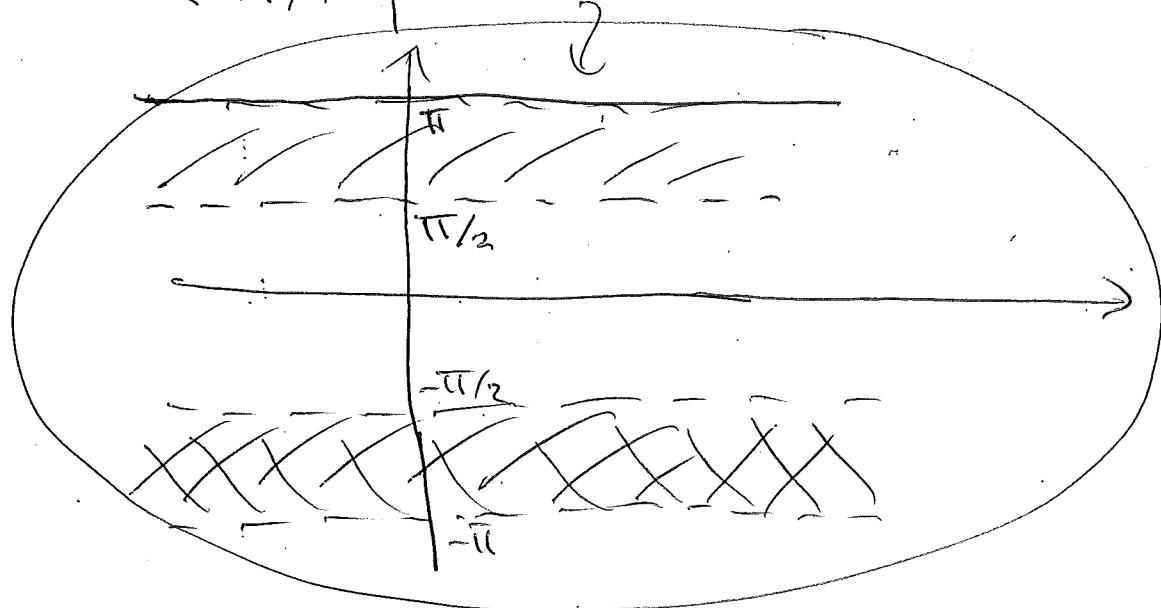
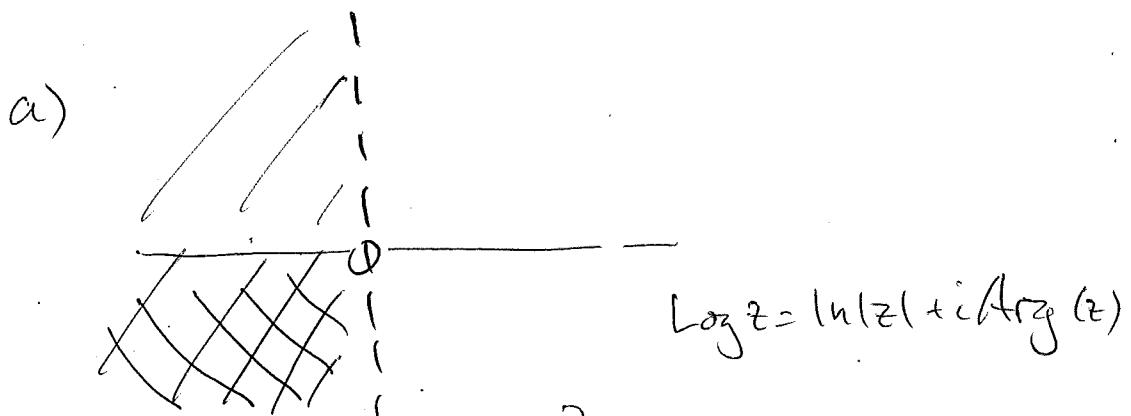
$$\Rightarrow \begin{cases} |z-2| = e \\ \arg(z-2) = \frac{\pi}{2} \end{cases}$$

$$\Rightarrow (z-2) = e e^{i\pi/2} = ei$$

$$\Rightarrow \boxed{z = 2 + ei}$$

[5] 3. a) Sketch the image of the half plane $\{Re(z) < 0\}$ under the map $\text{Log } z$ (principal branch of logarithm).

b) Sketch the image of the line $y = x$ under the map e^z .



[5] 4. Show that $p(z) = z^3 + 5z^2 + (1+i)$ has no roots in the 1st quadrant between the rays $\theta = 0$ and $\theta = \pi/6$

$$\begin{aligned} P(re^{i\theta}) &= r^3 e^{i3\theta} + 5r^2 e^{i2\theta} + 1 + i \\ &= (r^3 \cos 3\theta + 5r^2 \sin 2\theta + 1) \\ &\quad + i(r \cos 3\theta - 2r \sin 2\theta) \end{aligned}$$

$$0 < \theta < \frac{\pi}{6} \Rightarrow 0 < 2\theta, 3\theta < \frac{\pi}{2}$$

$$\Rightarrow 0 < \cos 2\theta, \cos 3\theta$$

$$\Rightarrow \operatorname{Re}(P(re^{i\theta})) > 0 \quad (\text{note: } r > 0)$$

\therefore no roots in above region

[6] 5.

Let $f(x+iy) = (x^2 + 2xy + y^2 + 3y) + i(2x^2 + 2y^2 + 3x)$.

(a) sketch the set of all points on the plane where f differentiable?

(b) where is f analytic?

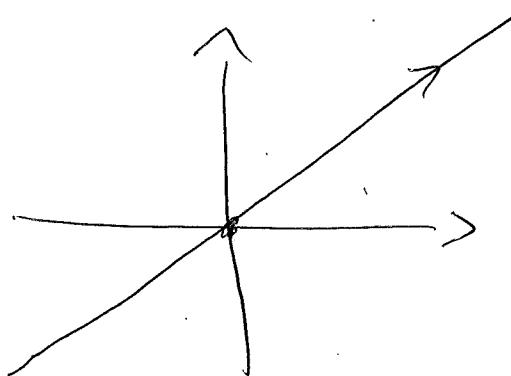
(c) find $f'(0)$.

$$\text{change to } -2x^2 + 2y^2 - 3x$$

$$\begin{aligned} \text{a) } \left\{ \begin{array}{l} u_x = 2x + 2y = V_y = 4y \\ u_y = 2x + 2y + 3 = -V_x = -(-4x - 3) \\ \quad \quad \quad = 4x + 3 \end{array} \right. \end{aligned}$$

$$\Leftrightarrow \left\{ \begin{array}{l} 2x = 2y \\ +2y = +2x \end{array} \right.$$

$$\Leftrightarrow x = y$$



b) nowhere

$$\text{c) } f'(0) = (u_x + iV_x)(0,0)$$

$$= 0 - i3$$

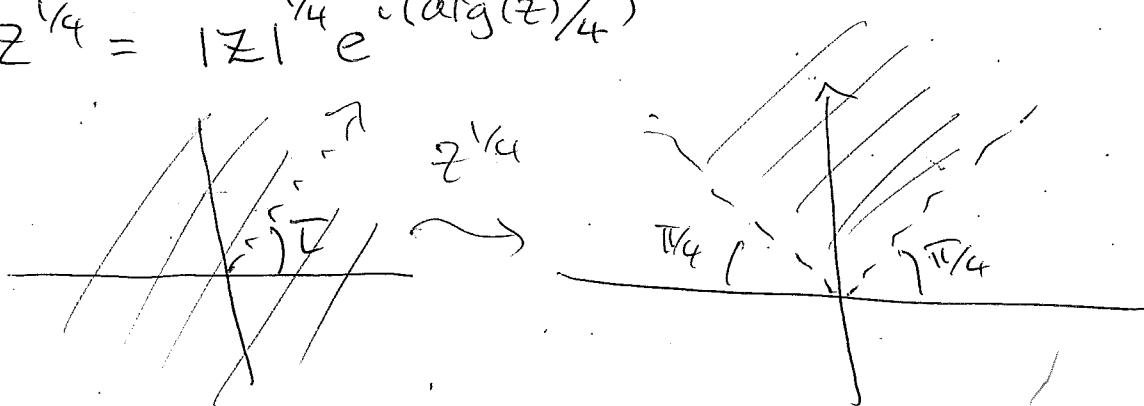
$$=-3i$$

[5] 6.

- a) Write the definition of the multivalued function $z^{1/4}$.
 b) Choose a branch of $z^{1/4}$ whose image fills the wedge between the lines $y = x$, $y = -x$ in the upper half plane.
 b) Find a real valued harmonic function $\phi(x, y)$ on the plane which is positive in the strip $(0 < y < 1)$ and equals to zero on the lines $y = 0$ and $y = 1$.

a) $z^{1/4} = |z|^{1/4} e^{i(\arg(z)/4)}$

b)



$$\Rightarrow \frac{\pi}{4} < \frac{\arg z}{4} \leq \frac{3\pi}{4}$$

$$\Rightarrow \begin{cases} \frac{\pi}{4} = \pi/4 \\ \frac{\pi + 2\pi}{4} = 3\pi/4 \end{cases}$$

$$\Rightarrow \tau = \pi$$

so take $\arg_{\pi}(z)$ in $z^{1/4}$

c) note: $e^{x+iy} = e^x \cos y + i e^x \sin y$

$$e^{\pi(x+iy)} = e^{\pi x} \cos(\pi y) + i e^{\pi x} \sin(\pi y)$$

$e^{\pi z}$ analytic on \mathbb{C}

$\therefore e^{\pi x} \sin(\pi y)$ harmonic on \mathbb{R}^2 , and has given properties

- [5] 7. Suppose the image of an entire function $f = u + iv$ is entirely contained on the unit circle in the plane.

Show that u is constant.

$$u^2 + v^2 = 1$$

$$\rightarrow \begin{cases} 2u u_x + 2v v_x = 0 \\ 2u u_y + 2v v_y = 0 \end{cases}$$

$$\left\{ \begin{array}{l} 2u u_x - 2v u_y = 0 \\ 2u u_y + 2v u_x = 0 \end{array} \right. \quad (\text{CR})$$

$$\rightarrow \begin{cases} 2u^2 u_x - 2uv u_y = 0 \\ 2uv u_y + 2v^2 u_x = 0 \end{cases}$$

$$\rightarrow \begin{cases} u^2 u_x + v^2 u_x = 0 \\ u_x(u^2 + v^2) = 0 \end{cases}$$

$$\therefore u_x = 0 \quad \text{on } \mathbb{R}^2$$

$$\rightarrow u_x(u^2 + v^2) = 0$$

$$\boxed{\therefore u_x = 0}$$

on \mathbb{R}^2

likewise, show

$$\boxed{u_y = 0} \quad \text{on } \mathbb{R}^2$$

$$\therefore \boxed{v_x, v_y = 0 \text{ by (CR)}} \quad \text{The End} \quad \text{on } \mathbb{R}^2$$

$\therefore u, v$ thus f are constant on \mathbb{C}