“Quadrature” means numerical integration. Main idea: find a polynomial interpolant as proxy/model for \( f(x) \) and integrate that instead. A reference is Chapter 7 of Süli and Mayers book.

**Setup:** given \( f(x_k) \) at \( n + 1 \) equally spaced points \( x_k = x_0 + k \cdot h \), \( k = 0, 1, \ldots, n \), where \( h = (x_n - x_0)/n \). Suppose that \( p_n(x) \) interpolates this data.

**Idea:** does

\[
\int_{x_0}^{x_n} f(x) \, dx \approx \int_{x_0}^{x_n} p_n(x) \, dx?
\]

We investigate the error in such an approximation below, but note that

\[
\int_{x_0}^{x_n} p_n(x) \, dx = \int_{x_0}^{x_n} \sum_{k=0}^{n} f(x_k) \cdot L_{n,k}(x) \, dx = \sum_{k=0}^{n} w_k f(x_k),
\]

where the coefficients

\[
w_k = \int_{x_0}^{x_n} L_{n,k}(x) \, dx
\]

\( k = 0, 1, \ldots, n \), are independent of \( f \). A formula

\[
\int_{a}^{b} f(x) \, dx \approx \sum_{k=0}^{n} w_k f(x_k)
\]

with \( x_k \in [a, b] \) and \( w_k \) independent of \( f \) for \( k = 0, 1, \ldots, n \) is called a quadrature formula; the coefficients \( w_k \) are known as weights. The specific form (1)–(3), based on equally spaced points, is called a Newton–Cotes formula of order \( n \).

**Trapezoidal Rule:** \( n = 1 \) (also known as the trapezium rule):

\[
\int_{x_0}^{x_1} f(x) \, dx \approx \frac{h}{2} [f(x_0) + f(x_1)]
\]

**Simpson’s Rule:** \( n = 2 \):

\[
\int_{x_0}^{x_2} f(x) \, dx \approx \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)]
\]

**Note:** The trapezoidal rule is exact if \( f \in \Pi_1 \), since if \( f \in \Pi_1 \implies p_1 = f \). Similarly, Simpson’s Rule is exact if \( f \in \Pi_2 \), since if \( f \in \Pi_2 \implies p_2 = f \). (In fact it is better, see
next page... The highest degree of polynomial exactly integrated by a quadrature rule is called the (polynomial) degree of accuracy (or degree of exactness).

**Error:** we can use the error in interpolation directly to obtain

\[
\int_{x_0}^{x_n} [f(x) - p_n(x)] \, dx = \int_{x_0}^{x_n} \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\xi(x)) \, dx
\]

so that

\[
\left| \int_{x_0}^{x_n} [f(x) - p_n(x)] \, dx \right| \leq \frac{1}{(n+1)!} \max_{\xi \in [x_0,x_n]} |f^{(n+1)}(\xi)| \int_{x_0}^{x_n} |\pi(x)| \, dx,
\]

which, e.g., for the trapezoidal rule, \( n = 1 \), gives

\[
\left| \int_{x_0}^{x_1} f(x) \, dx - \frac{(x_1 - x_0)}{2} [f(x_0) + f(x_1)] \right| \leq \frac{(x_1 - x_0)^3}{12} \max_{\xi \in [x_0,x_1]} |f''(\xi)|.
\]

In fact, we can prove a tighter result using the Integral Mean-Value Theorem\(^1\):

**Theorem.** \( \int_{x_0}^{x_1} f(x) \, dx - \frac{(x_1 - x_0)}{2} [f(x_0) + f(x_1)] = -\frac{(x_1 - x_0)^3}{12} f''(\xi) \) for some \( \xi \in (x_0, x_1) \).

For \( n > 1 \), (4) gives pessimistic bounds. But one can prove better results such as:

**Theorem.** Error in Simpson’s Rule I: if \( f''' \) is continuous on \((x_0, x_2)\), then

\[
\left| \int_{x_0}^{x_2} f(x) \, dx - \frac{(x_2 - x_0)}{6} [f(x_0) + 4f(x_1) + f(x_2)] \right| \leq \frac{(x_2 - x_0)^5}{720} \max_{\xi \in [x_0,x_2]} |f'''(\xi)|.
\]

**Proof.** First a more general lemma:

**Lemma.** Let \( x_0 = x_1 - h \) and \( x_2 = x_1 + h \). Suppose \( f''' \) is continuous on \((x_0, x_2)\). Then

\[
\frac{f(x_0) - 2f(x_1) + f(x_2)}{h^2} = f''(x_1) + \frac{h}{12} f'''(\eta),
\]

(5)

Proof uses Taylor expansions and Intermediate-Value Theorem\(^2\) for some \( \eta \in (x_0, x_2) \). \( \Box \)

Back to the main proof: now for any \( x \in [x_0, x_2] \), we may use Taylor’s Theorem again

\[^1\text{Integral Mean-Value Theorem: if } f \text{ and } g \text{ are continuous on } [a, b] \text{ and } g(x) \geq 0 \text{ on this interval, then there exists an } \eta \in (a, b) \text{ for which } \int_a^b f(x)g(x) \, dx = f(\eta) \int_a^b g(x) \, dx.\]

\[^2\text{Intermediate-Value Theorem: if } f \text{ is continuous on a closed interval } [a, b], \text{ and } c \text{ is any number between } f(a) \text{ and } f(b) \text{ inclusive, then there is at least one number } \xi \text{ in the closed interval such that } f(\xi) = c. \text{ In particular, since } c = (df(a) + ef(b))/(d+e) \text{ lies between } f(a) \text{ and } f(b) \text{ for any positive } d \text{ and } e, \text{ there is a value } \xi \text{ in the closed interval for which } d \cdot f(a) + e \cdot f(b) = (d + e) \cdot f(\xi).\]
to deduce
\[
\int_{x_0}^{x_2} f(x) \, dx = f(x_1) \int_{x_1-h}^{x_1+h} \, dx + f'(x_1) \int_{x_1-h}^{x_1+h} (x - x_1) \, dx \\
+ \frac{1}{2} f''(x_1) \int_{x_1-h}^{x_1+h} (x - x_1)^2 \, dx + \frac{1}{6} f'''(x_1) \int_{x_1-h}^{x_1+h} (x - x_1)^3 \, dx \\
+ \frac{1}{24} \int_{x_1-h}^{x_1+h} f''''(\eta_1(x)) (x - x_1)^4 \, dx \\
= 2hf(x_1) + \frac{1}{2} h^3 f''(x_1) + \frac{1}{6} h^5 f'''(\eta_2) \\
= \frac{1}{2} h \left[ f(x_0) + 4f(x_1) + f(x_2) \right] + \frac{1}{6} h^5 f'''(\eta_2) - \frac{1}{36} h^5 f'''(\xi_3) \\
= \int_{x_0}^{x_2} p_2(x) \, dx + \frac{1}{180} \left( \frac{x_2 - x_0}{2} \right)^5 \left( 3 f'''(\eta_2) - 5 f'''(\xi_3) \right)
\]

where \( \eta_1(x) \) and \( \eta_2 \in (x_0, x_2) \), using the Integral Mean-Value Theorem and the lemma. Thus, taking moduli,

\[
\left| \int_{x_0}^{x_2} [f(x) - p_2(x)] \, dx \right| \leq \frac{8}{25} \cdot \frac{1}{180} (x_2 - x_0)^5 \max_{\xi \in [x_0, x_2]} |f'''(\xi)|
\]

as required. \( \Box \)

**Note:** Simpson’s Rule is exact if \( f \in \Pi_3 \) since then \( f''' \equiv 0 \).

In fact, it is possible to compute a slightly stronger bound.

**Theorem.** Error in Simpson’s Rule II: if \( f''' \) is continuous on \( (x_0, x_2) \), then

\[
\int_{x_0}^{x_2} f(x) \, dx = \frac{x_2 - x_0}{6} \left[ f(x_0) + 4f(x_1) + f(x_2) \right] - \frac{(x_2 - x_0)^5}{2880} f'''(\xi)
\]

for some \( \xi \in (x_0, x_2) \).

**Proof.** See Süli and Mayers, Thm. 7.2. \( \Box \)