Math 253 Notes on Moments of Inertia

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1 Moments of Inertia

We’ve previously seen moments when calculating centre of mass of a lamina. This involved two double integrals:

\[ M_y = \int \int_D x \rho(x,y) \, dA \]
\[ M_x = \int \int_D y \rho(x,y) \, dA \]

These can also be called the “first moments”; here we look at the "second moments" or "moments of inertia".

1.1 Kinetic Energy of a spinning lamina

Suppose our lamina (which lies in the x-y plane) is rotating around the z-axis (note this is orthogonal to the lamina) at a constant angular rotational speed \( \omega \) radians/s. (E.g., 60 rpm = 1 rev/s = \( 2\pi \) rad/s). Find the Kinetic Energy of the lamina. [Draw diagram!]

Riemann sum idea: as before, we consider a small rectangular piece \( R_{ij} \) with area \( \Delta x \Delta y \). The kinetic energy of a point mass is \( \frac{1}{2}mv^2 \). Its going to be small in the limit so we use this to get:

\[ \frac{1}{2} \rho(x_i,y_j) \Delta x \Delta y |\vec{v}_{ij}|^2. \]

The piece \( R_{ij} \) moves faster the further it is from the axis of rotation (z-axis, \( (x,y) = (0,0) \)). Different pieces move at different speeds. Our piece has kinetic energy:

\[ \frac{1}{2} \rho(x_i,y_j) \Delta x \Delta y \omega^2 \left( x_i^2 + y_i^2 \right). \]

So take the Riemann sum over all pieces of the lamina and we get:

\[ K = \frac{1}{2} \omega^2 \int \int_D (x^2 + y^2) \rho(x,y) \, dA. \]

We define \( I_0 \) the moment of inertia about the z-axis as just the integral part:

\[ I_0 = \int \int_D (x^2 + y^2) \rho(x,y) \, dA. \]

Larger \( I_0 \) means more energy (work) to rotate the lamina about the z-axis.
1.2 About some other axis?
A similar argument shows how to compute the moment of inertia about some other axis parallel to the $z$-axis, centred at $(x, y) = (a, b)$:

$$I_0 = \int \int_D ((x - a)^2 + (y - b)^2)\rho(x, y)dA.$$  

And in particular about the centre of mass $(x, y) = (\bar{x}, \bar{y})$, this would be:

$$I_{0,c} =$$

1.3 Rotation around $x$ or $y$ axes
What about rotating around the $x$-axis and $y$-axis? This gives the moment of inertia about the $y$-axis denoted $I_y$ and the moment of inertia about the $x$-axis denoted $I_x$. [Draw diagrams]

$$I_y =$$

$$I_x =$$

Note relationship to previous,

$$I_0 =$$

1.4 Changing the axis of rotation
Suppose we have $I_{0,c}$ and want rotation around $z$-axis? Let $M$ be overall mass of lamina. We get:

$$I_0 =$$

1.5 Examples
1. Find moment of inertia about the $z$-axis of a uniform circular disc of radius $R$ and total mass $M$, centred at the origin.

2. Find same, but with disc centred at point $(a, b)$.

3. Find same, for a uniform rectangular plate, mass $M$, axis through centre, size $a \times b$. 