Numerical Methods for Differential Equations: Homework 5

Due by 2pm Tuesday 24th November. ***Please submit your hardcopy at the start of lecture.*** *The answers you submit should be attractive, brief, complete, and should include program listings and plots where appropriate. The use of "publish" in* MATLAB *is one possible approach.*

Problem 1: von Neumann analysis Perform a von Neumann stability analysis for the discretization of the problem $u_t + au_x$ using forward Euler with each of these spatial discretizations:

1. first-order forward difference for u_x.
2. first-order backward difference for u_x.
3. second-order centered difference for u_x.

Do the algebra in terms of the "*Courant number* $\nu = \frac{ka}{h}$" where k is the time-step and h is the spatial step. For each case:

- draw a picture of the complex plane with a unit circle centered at the origin;
- sketch the set of $g(\xi)$ as ξ varies;
- discuss whether its possible for $|g(\xi)| \leq 1$ and if so, under what restrictions on k?

You might want to look at [1] or another book on von Neumann analysis for details.

Problem 2: Stability Write a code to solve $u_t + au_x$ on a periodic domain with a non-zero initial condition of your choice. For a spatial discretization, use:

1. first-order forward difference for u_x.
2. first-order backward difference for u_x.
3. second-order centered difference for u_x.

Compare the stability of the results with what your analysis in Problem 1.

Problem 3: Gray–Scott pattern formation in 1D The Gray–Scott equations are a pair of coupled reaction-diffusion equations that lead to remarkable patterns (See Pearson, *Science* 1993, and search online for "xmorphia".) The PDEs are:

$$u_t = \varepsilon_u \Delta u - uv^2 + F(1-u), \qquad v_t = \varepsilon_v \Delta v + uv^2 - (c+F)v,$$

where u and v are functions of space and time. For this question, try $c = .065$ and $F = .06$.

First, set the diffusion constants to zero. Solve the resulting two ODEs numerically with the forward Euler method and determine (experimentally or otherwise) the time-step restriction on k. Try various initial conditions, what steady state solutions do you observe? This is known as the *zero-diffusion steady state.*

Usually we think of diffusion as a process that smooths and stabilizes: not so here. Alan Turing in 1952 [2] proposed what is now known as the *Turing Instability* (he also proposed it as a possible mechanism for animal coat pattern formation: still a topic of current research). To investigate this phenomenon on the G–S equations, take these values for the diffusion constants: $\varepsilon_u = 6 \times 10^{-5}$ and $\varepsilon_v = 2 \times 10^{-5}$. Now write a code to solve the G–S equations on $-1 \leq x < 1$ with periodic boundary conditions. Use an initial condition like:

```
>> v = (abs(x-0.1)<.1)  +  0.05*randn(size(x));
>> u = 1 - v;
```

Based on your answer for the ODEs and what you know about the heat equation, can you guess a formula for a reasonable time-step restriction?

Problem 4: Gray–Scott in 2D Reconsider the above equations but this time u and v are functions of x, y, t. We will consider two sets of parameter values:

$$\text{(I) } c = .065, F = .06, \qquad \text{(II) } c = .065, F = .03,$$

(and use ε as above). Write a program to solve the G–S equations on a $M \times M$ regular grid in the square $0 \leq x, y \leq 1$ with periodic boundary conditions. For initial conditions take

$$u(x, y) = \min\{1, 10\sqrt{(x - .2)^2 + (y - .2)^2}\}, \quad v(x, y) = \max\{0, 1 - 10\sqrt{(x - .3)^2 + 2(y - .3)^2}\};$$

u and v are plotted above. It is up to you whether your code is implicit or explicit.

Produce four plots of $u(x, y)$ corresponding to parameters (I) and (II) and times $t = 500$ and $t = 1000$. Use `contourf` or `pcolor` to make the plots.

Next compute four numbers: the values $u(0.75, 0.75)$ for the four cases above. By experimenting with various grid resolutions and time steps, can you obtain numbers that you believe are correct to, say, 2 significant digits? (Try not to warm up the planet too much while doing this. . .)

References

[1] R. J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems.* SIAM, 2007.

[2] A. M. Turing. The chemical basis of morphogenesis. *Phil. Trans. Roy. Soc. Lond.*, B237:37–72, 1952.