Numerical Methods for Differential Equations: Homework 5

Due by 2pm Tuesday 24th November. Please submit your hardcopy at the start of lecture. The answers you submit should be attractive, brief, complete, and should include program listings and plots where appropriate. The use of “publish” in MATLAB is one possible approach.

Problem 1: von Neumann analysis Perform a von Neumann stability analysis for the discretization of the problem \( u_t + au_x \) using forward Euler with each of these spatial discretizations:

1. first-order forward difference for \( u_x \).
2. first-order backward difference for \( u_x \).
3. second-order centered difference for \( u_x \).

Do the algebra in terms of the “Courant number \( \nu = \frac{kh}{k} \)” where \( k \) is the time-step and \( h \) is the spatial step. For each case:

- draw a picture of the complex plane with a unit circle centered at the origin;
- sketch the set of \( g(\xi) \) as \( \xi \) varies;
- discuss whether it’s possible for \( |g(\xi)| \leq 1 \) and if so, under what restrictions on \( k \)?


Problem 2: Stability Write a code to solve \( u_t + au_x \) on a periodic domain with a non-zero initial condition of your choice. For a spatial discretization, use:

1. first-order forward difference for \( u_x \).
2. first-order backward difference for \( u_x \).
3. second-order centered difference for \( u_x \).

Compare the stability of the results with what your analysis in Problem 1.

Problem 3: Gray–Scott pattern formation in 1D The Gray–Scott equations are a pair of coupled reaction-diffusion equations that lead to remarkable patterns (See Pearson, Science 1993, and search online for “xmorphia”).

The PDEs are:

\[
\begin{align*}
    u_t &= \varepsilon_u \Delta u - uv^2 + F(1-u), \\
    v_t &= \varepsilon_v \Delta v + uv^2 - (c+F)v,
\end{align*}
\]

where \( u \) and \( v \) are functions of space and time. For this question, try \( c = 0.065 \) and \( F = 0.06 \).

First, set the diffusion constants to zero. Solve the resulting two ODEs numerically with the forward Euler method and determine (experimentally or otherwise) the time-step restriction on \( k \). Try various initial conditions, what steady state solutions do you observe? This is known as the zero-diffusion steady state.

Usually we think of diffusion as a process that smooths and stabilizes: not so here. Alan Turing in 1952 [2] proposed what is now known as the Turing Instability (he also proposed it as a possible mechanism for animal coat pattern formation: still a topic of current research). To investigate this phenomenon on the G–S equations, take these values for the diffusion constants: \( \varepsilon_u = 6 \times 10^{-5} \) and \( \varepsilon_v = 2 \times 10^{-5} \). Now write a code to solve the G–S equations on \(-1 \leq x < 1\) with periodic boundary conditions. Use an initial condition like:

\[
\begin{align*}
    v &= (\text{abs}(x-0.1)<0.1) + 0.05*\text{randn(size(x))}; \\
    u &= 1 - v;
\end{align*}
\]

Based on your answer for the ODEs and what you know about the heat equation, can you guess a formula for a reasonable time-step restriction?
Problem 4: Gray–Scott in 2D  
Reconsider the above equations but this time $u$ and $v$ are functions of $x, y, t$. 
We will consider two sets of parameter values:

(I) $c = .065, F = .06$,  
(II) $c = .065, F = .03$,  

(and use $\varepsilon$ as above). Write a program to solve the G–S equations on a $M \times M$ regular grid in the square $0 \leq x, y \leq 1$ with periodic boundary conditions. For initial conditions take

$u(x, y) = \min\{1, 10\sqrt{(x-.2)^2 + (y-.2)^2}\},$  
$v(x, y) = \max\{0, 1 - 10\sqrt{(x-.3)^2 + 2(y-.3)^2}\};$

$u$ and $v$ are plotted above. It is up to you whether your code is implicit or explicit.

Produce four plots of $u(x, y)$ corresponding to parameters (I) and (II) and times $t = 500$ and $t = 1000$. Use \texttt{contourf} or \texttt{pcolor} to make the plots.

Next compute four numbers: the values $u(0.75, 0.75)$ for the four cases above. By experimenting with various grid resolutions and time steps, can you obtain numbers that you believe are correct to, say, 2 significant digits? (Try not to warm up the planet too much while doing this...)

References
