Numerical Methods for Differential Equations
Homework 3

Due by 2pm Thursday 29 October. ***Please submit your hardcopy at the start of lecture.*** *The answers you submit should be attractive, brief, complete, and should include program listings and plots where appropriate. The use of "publish" in* MATLAB *is one possible approach.*

Problem 1 (Use of `ode113` and `ode45`, problem from [2].) Consider the third-order initial value problem

$$\begin{aligned} &v'''(t) + v''(t) + 4v'(t) + 4v(t) = 4t^2 + 8t - 10, \\ &v(0) = -3, \quad v'(0) = -2, \quad v''(0) = 2. \end{aligned}$$

(a) Verify that the function $v(t) = -\sin(2t) + t^2 - 3$ is a solution. Do this either by hand or using symbolic computation (e.g., `syms t` in the MATLAB symbolic toolbox.)

(b) Rewrite this problem as a first order system of the form $u'(t) = f(u(t), t)$ where $u(t) \in \mathbb{R}^3$. Make sure you also specify the initial condition $u(0) = \eta$ as a 3-vector.

(c) Use the MATLAB function `ode113` to solve this problem over the time interval $0 \le t \le 2$. Plot the true and computed solutions to make sure you've done this correctly.

(d) Test the MATLAB solver by specifying different tolerances spanning several orders of magnitude. Create a table showing the maximum error in the computed solution for each tolerance and the number of function evaluations required to achieve this accuracy.

(e) Repeat part (d) using the MATLAB function `ode45`, which uses an embedded pair of Runge-Kutta methods instead of Adams-Bashforth-Moulton methods.

If using Python, use `scipy.integrate.odeint`. For the last part, use `scipy.integrate.ode` with `set_integrator('dopri5')`.

Problem 2 (A three-body problem from [1, Ch. II, pg 129].) Earth is near the origin at $(-\mu, 0)$, the moon is at $(1-\mu, 0)$ and we're flying around in a small spacecraft at point $(y_1(t), y_2(t))$. Here are the equations:

$$\begin{aligned}
y_1'' &= y_1 + 2y_2' - \bar{\mu}\frac{y_1+\mu}{D_1} - \mu\frac{y_1-\bar{\mu}}{D_2}, \\
y_2'' &= y_2 - 2y_1' - \bar{\mu}\frac{y_2}{D_1} - \mu\frac{y_2}{D_2}, \\
\text{where} \quad D_1 &= ((y_1+\mu)^2 + y_2^2)^{3/2}, \\
D_2 &= ((y_1-\bar{\mu})^2 + y_2^2)^{3/2},
\end{aligned}$$

with parameters

$$\begin{aligned}
\mu = 0.012277471,\ \bar{\mu} = 1-\mu,\ y_1(0) &= 0.994,\ y_1'(0) = 0,\ y_2(0) = 0, \\
y_2'(0) &= -2.00158510637908252240537862224, \\
t_{\text{end}} &= 17.0652165601579625588917206249.
\end{aligned}$$

Write a code to solve this with the forward Euler method from $t = 0$ to $t = t_{\text{end}}$. Use 24000 timesteps. Plot the solution in the $y_1(t), y_2(t)$ plane. Repeat with `ode45()` and plot on the same axis.

Which one seems to give you a closed orbit and which one sends you careening off into space? Which result do you trust to be correct and why?

Problem 3 Three unit masses are fixed at positions $(1, 0)$, $(\cos 120°, \sin 120°)$ and $(\cos 240°, \sin 240°)$ in the plane. Another unit mass p starts motionless at $(2, -2)$ at $t = 0$ and then moves around freely under the influence of inverse-square forces attracting it to the three fixed masses. Where is p at $t = 40$?

Use Matlab's `ode113` to solve this numerically as an ODE problem. In addition to determining the required pair of numbers, make plots of the orbit in the plane and of $\theta(t)$, the angle of the particle with respect to the origin as a function of t.

(*Hint:* be careful to compute your trajectory to enough precision that it comes out correct to at least several digits of precision. To adjust error tolerances for `ode113`, try `help odeset`.)

References

[1] E. Hairer, S. P. Nørsett, and G. Wanner. *Solving ordinary differential equations I: Nonstiff problems.* Springer-Verlag, second edition, 1993.

[2] R. J. LeVeque. *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems.* SIAM, 2007.