14: Finite Element Method

Weak/Variational form of the PDE

Given PDE in $u$. E.g., $-u'' = f$, $x \in (a,b)$.

- multiple by test function $v$
- integration by parts. (move derivatives onto $v$).

Notation: inner product: $(v,w) = \int_a^b v(x)w(x)dx$.

[Example: Poisson.]

PDE $\iff$ Weak form
Find $u$ solving PDE $\iff$ find $u$ such that weak form satisfied for all $v$.

Take test functions $v$ from $V$: the set of continuous functions $v$ on $[a,b]$ with $v'$ piecewise continuous and bounded on $[a,b]$ and $v(a) = v(b) = 0$.

Variational problem (V):
Find $u \in V$ such that $(u',v') = (f,v)$ for all $v \in V$.
[More generally, with a bilinear form $a(u,v)$. For Poisson this is $a(u,v) = (u',v')$.]

Minimization problem (M):
Find $u \in V$ such that $F(u) \leq F(v)$ for all $v \in V$, where $F(v) = \frac{1}{2} (v',v') - (f,v)$.
Can show that (V) and (M) and “classical” Poisson problem have same solution.

1D Finite Element Discretization

Continuing Poisson example. Use $[0,1]$. [see also Johnson pg 19]

The grid
$0 = x_0 < x_1 < \ldots < x_M < x_{M+1} = 1$. Intervals $I_j = (x_{j-1},x_j)$ of length $h_i = x_j - x_{j-1}$. Set $h = \max h_j$.

Let $V_h$ be the subset of $V$ such that $v \in V_h$ is linear on each subinterval $I_j$. [Draw example]

Basis functions
Choose $\phi_j(x)$ as the “hat function”. 1 at $x_j$, 0 at others. [sketch].
Then represent $v$ as sum of these. Note this makes the $\phi_j$ a basis for $V_h$.
[note as usual, have to worry about end points, here (Dirichlet) we don’t have to worry too much.]
Write $u_h$ as sum of basis, then take $v$ as the basis elements one-at-a-time.
Linear algebra

\[ A \xi = b \]

\( A \): stiffness matrix: \( a_{ij} = (\phi'_i, \phi'_j) \).
\( b \): load vector: \( b_i = (f, \phi_i) \).

Similar to FD (for this grid anyway). Note RHS is different (in FD it was point samples).

note: \( 1/h \) not \( 1/h^2 \) like in finite differences.

References


FEM for the Poisson equation

Consider \(-\nabla^2 u = f\), in a domain \( \Omega \) with \( u = 0 \) on boundary \( \Gamma \).

Now weak form is:

\[ a(u, v) = (f, v) \]

with bilinear form: \( a(u, v) = \int_{\Omega} \nabla u \nabla v \, dx \). Different problems have different bilinear forms.

Triangulations

Consider \( \Omega \) with polygonal boundary and subdivide into non-overlapping triangles \( K_i \). [diagram]

Vertices of the triangles also known as nodes.

Basis functions on the triangulation: as above, but now “tent functions”, 1 at node \( j \), 0 at neighbours (and all others). [diagram]

In principle, can assemble matrix from this. In my humble opinion this is not a job for a human being: many software projects that automate this, FEniCS for example.

More on Basis functions

We’ve looked at linear basis functions. These are “P1” elements. You can use higher degree polynomials, giving quadratic “P2” elements. These are still continuous across triangle boundaries.

DGFEM: Discontinuous Galerkin drops the continuity assumption: typically high-degree polynomials over each triangle, but no further continuity.

What is the “element” in finite element method?

“We define a finite element to mean a triple \((K, P_K, \Sigma)\).” [section 3.3 of Johnson]. For example: \( K \): a triangle, \( P_K \): polynomial over \( K \), \( \Sigma \): values at the vertices of \( K \).