
14: Finite Element Method

Weak/Variational form of the PDE

Given PDE in u . E.g., $-u'' = f, x \in (a, b)$.

- multiply by test function v
- integration by parts. (move derivatives onto v).

Notation: inner product: $(v, w) = \int_a^b v(x)w(x)dx$.

[Example: Poisson.]

PDE \iff Weak form

Find u solving PDE \iff find u such that weak form satisfied for all v .

Take test functions v from V : the set of continuous functions v on $[a, b]$ with v' piecewise continuous and bounded on $[a, b]$ and $v(a) = v(b) = 0$.

Variational problem (V):

Find $u \in V$ such that $(u', v') = (f, v)$ for all $v \in V$.

[More generally, with a *bilinear form* $a(u, v)$. For Poisson this is $a(u, v) = (u', v')$.]

Minimization problem (M):

Find $u \in V$ such that $F(u) \leq F(v)$ for all $v \in V$, where $F(v) = \frac{1}{2}(v', v') - (f, v)$.

Can show that (V) and (M) and “classical” Poisson problem have same solution.

1D Finite Element Discretization

Continuing Poisson example. Use $[0, 1]$. [see also Johnson pg 19]

The grid

$0 = x_0 < x_1 < \dots < x_M < x_{M+1} = 1$. Intervals $I_j = (x_{j-1}, x_j)$ of length $h_j = x_j - x_{j-1}$. Set $h = \max h_j$.

Let V_h be the subset of V such that $v \in V_h$ is linear on each subinterval I_j . [Draw example]

Basis functions

Choose $\phi_j(x)$ as the “hat function”. 1 at x_j , 0 at others. [sketch].

Then represent v as sum of these. Note this makes the ϕ_j a basis for V_h .

[note as usual, have to worry about end points, here (Dirichlet) we don't have to worry too much.]

Write u_h as sum of basis, then take v as the basis elements one-at-a-time.

Linear algebra

$$A\xi = b$$

A: stiffness matrix: $a_{ij} = (\phi'_i, \phi'_j)$.

b load vector: $b_i = (f, \phi_i)$

Similar to FD (for this grid anyway). Note RHS is different (in FD it was point samples).

note: $1/h$ not $1/h^2$ like in finite differences.

References

Claes Johnson, Numerical Solution of Partial Differential Equations by the Finite Element Method, Cambridge Univ Press 1987.

Endre Suli, Lecture Notes on Finite Element Methods for Partial Differential Equations.

FEM for the Poisson equation

Consider $-\nabla^2 u = f$, in a domain Ω with $u = 0$ on boundary Γ .

Now weak form is:

$$a(u, v) = (f, v)$$

with *bilinear form*: $a(u, v) = \int_{\Omega} \nabla u \nabla v dx$. Different problems have different bilinear forms.

Triangulations

Consider Ω with polygonal boundary and subdivide into non-overlapping triangles K_i . [diagram]

Vertices of the triangles also known as nodes.

Basis functions on the triangulation: as above, but now “tent functions”, 1 at node j , 0 at neighbours (and all others). [diagram]

In principle, can assemble matrix from this. In my humble opinion this is not a job for a human being: many software projects that automate this, FEniCS for example.

More on Basis functions

We’ve looked at linear basis functions. These are “P1” elements. You can use higher degree polynomials, giving quadratic “P2” elements. These are still continuous across triangle boundaries.

DGFEM: Discontinuous Galerkin drops the continuity assumption: typically high-degree polynomials over each triangle, but no further continuity.

What is the “element” in finite element method?

“We define a finite element to mean a triple (K, P_K, Σ) .” [section 3.3 of Johnson]. For example: K : a triangle, P_K : polynomial over K , Σ : values at the vertices of K .