The following questions and problems came up naturally during class. They are not necessarily difficult but they should help understand the material from class.

(1) Recall the correspondence
\[ M(k; N) \xleftarrow{\pi_1} M(k, k + r; N) \xrightarrow{\pi_2} M(k + r; N) \]
from class. We defined
\[ e(r)(\cdot) := \pi_2^* \pi_1^*(\cdot) : Fun(M(k; N)) \to Fun(M(k + r; N)) \]
and likewise \( f(r)(\cdot) := \pi_1^* \pi_2^*(\cdot) \). Check that \( e(r) = r! e(r) \) and \( f(r) = r! f(r) \).

(2) In the quantized setup
\[ M_q(k; N) \xleftarrow{\pi_1} M_q(k, k + r; N) \xrightarrow{\pi_2} M_q(k + r; N) \]
with \( e(r)(\cdot) := q^{-r(N-k-r)} \pi_2^* \pi_1^*(\cdot) \) and \( f(r)(\cdot) = q^{-rk} \pi_1^* \pi_2^*(\cdot) \) show that \( e(r) = \lceil r \rceil q^r e(r) \) and \( f(r) = \lceil r \rceil q^r f(r) \).

(3) Explain why \( \delta \Delta \in Fun(X \times X) \) is the kernel which induces the identity map on \( Fun(X) \) (here \( \Delta \subset X \times X \) is the diagonal).

(4) Recall the map \( t_q : Fun(M_q(k, N)) \to Fun(M_q(N - k, N)) \) given by
\[ t_q(\delta_S) = \sum S \cap S' = 0 \delta_{S'} \]
for \( S \) a subspace of rank \( k \). Find an “elementary” argument that shows it is an equivalence (I don’t think I know one).

(5) Show that
\[ \sum_{r \geq 0} (-1)^r q^{r(r-1)} q^{r(m-r)} [^m_r]_q = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases} \]

(6) Show that
\[ \sum_{r \geq 0} (-1)^r q^{r(r+1)} q^{r(m-r)} [^m_r]_q = (1 - q^2)(1 - q^4) \ldots (1 - q^{2(m-1)}) \ldots \]

(7) Can you show the inverse of \( t_q \) from (1) is given by
\[ \delta_S \mapsto \sum_m \sum_{rk(S \cap S') = m} (1 - q^2)(1 - q^4) \ldots (1 - q^{2(m-1)}) \delta_{S'} \]
by some “elementary” or reasonably direct argument? (I don’t know how.)
(8) Prove the following cancellation lemma. Let $X, Y, Z, W, U, V$ be six objects in an additive category and consider a complex

$$\cdots \to U \overset{u}{\to} X \oplus Y \overset{f}{\to} Z \oplus W \overset{v}{\to} V \to \cdots$$

where $f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $u, v$ are arbitrary morphisms. If $D : Y \to W$ is an isomorphism, then (2) is homotopic to a complex

$$\cdots \to U \overset{u}{\to} X \overset{A - B D^{-1} C}{\to} Z \overset{v \mid Z}{\to} V \to \cdots$$

(9) For a morphism $f : F \to G$ of sheaves check that the presheaf ker$(f)$ is actually a sheaf. Find an example where coker$(f)$ is just a presheaf (before sheafifying).

(10) Suppose $i : p \to X$ is the inclusion of a point in a topological space $X$ and $j : U := X \setminus \{p\} \to X$ is the inclusion of its complement. Show (from definition) that we have a short exact sequence

$$0 \to j_! \mathbb{C}_U \to \mathbb{C}_X \to i_* \mathbb{C}_p \to 0.$$

(11) For a sheaf $\mathcal{F}$ on a topological space $X$ show that $\text{Hom}_X(\mathbb{C}_X, \mathcal{F}) = \Gamma(\mathcal{F})$ (the space of global sections).

(12) If $\mathcal{C}$ is an abelian category and $S$ is the class of quasi-isomorphisms in $\text{Kom}(\mathcal{C})$ show that $S$ is localizable.

(13) Give an example of an abelian category $\mathcal{C}$ and $A \in \mathcal{C}$ such that $\text{Hom}_\mathcal{C}(A, \cdot)$ is not an exact functor (check that it is in fact not exact).

(14) Show that the Cech differential $d : C^p([U_i], \mathcal{F}) \to C^{p+1}([U_i], \mathcal{F})$ defined in class satisfies $d^2 = 0$.

(15) Consider $S^2 = U \cup V$ where $U = S^2 \setminus \{0\}$ and $V = S^2 \setminus \{\infty\}$. Compute the Cech cohomology $\check{H}^*(\{U, V\}, \mathbb{C}_{S^2})$. If you don’t get the same thing as the singular cohomology of $S^2$ find another cover of $S^2$ which does recover

$$\check{H}^*_{\text{sing}}(S^2, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } \ast = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

(16) Explain why $H^i(X, \mathcal{F}) \cong H^i(X, \mathcal{C}^*(\mathcal{F}))$ where $\mathcal{C}^*(\mathcal{F})$ is the Cech resolution of $\mathcal{F}$ (this is just an exercise in understanding fundamental concepts, you don’t need to prove that $\mathcal{C}^*(\mathcal{F})$ is in fact a resolution of $\mathcal{F}$).

(17) Compute $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ for $i \geq 0$ (hint: use a cover by two open subsets). If you are familiar with line bundles you can also try to compute $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))$ for $k = -1, -2$ (using the same cover).

(18) If $\mathcal{L}$ is a local system on $X$ with fiber $V$ show that $\Gamma(\mathcal{L}) = V^{\pi_1(X)}$.

(19) Recall that $\mathcal{L}_k (k \geq 1)$ denotes the rank one local system on $S^1$ which has monodromy $\exp(2\pi i/k)$. Show that $H^*(S^1, \mathcal{L}_k) = 0$ for any $\ast$ and $k \geq 2$. Hint: show that $\pi_{k\ast}(\mathbb{C}_Y) \cong \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_k$ where $\pi_k : Y \to S^1$ is the connected $k : 1$ cover.
(20) Suppose \( \mathcal{A} \) is a sheaf of algebras. Show that any \( \mathcal{A} \)-module \( \mathcal{M} \) is fine if \( \mathcal{A} \) is fine. Use this to show that if \( V \) is a smooth vector bundle on a smooth manifold \( X \) (and \( \mathcal{V} \) is the corresponding sheaf given by its sections) then \( H^i(X, \mathcal{V}) = 0 \) for \( i > 0 \).