PRESENTATIONS: ALGEBRAIC GEOMETRY II

(1) Recall the construction (from class) of maps \( f : X \to \mathbb{P}^n \) as given by a line bundle and a \( n + 1 \) sections which generate it. Such a map induces two functors \( f_* \) and \( f^* \) between the categories of coherent sheaves on each side.

Recently, in Section 4 of [GNR], a novel generalization of this construction is described. The variety \( X \) is replaced by a monoidal category \( \mathcal{C} \) (corresponding to the category \( \text{Coh}(X) \)) and the map \( f \) by two functors \( \iota_* \) and \( \iota^* \) to \( \text{Coh}(\mathbb{P}^n) \).

Explain and discuss their construction.

(2) Discuss the idea of Weyl, Cartier divisors and linear equivalence. How are they related to line bundles? Illustrate the difference with an example of a Weyl divisor which is not Cartier. What is the space of all divisors (can you describe it in terms of Čech cohomology)?

(3) Show that all line bundles on \( \mathbb{P}^n \) are of the form \( \mathcal{O}_{\mathbb{P}^n}(\ell) \) for some \( \ell \in \mathbb{Z} \). This requires thinking about the vanishing loci of sections and is therefore related to the topic above. Show that \( \text{Aut}(\mathbb{A}^n) \) is given by the group of affine transformations on \( \mathbb{A}^n \). Hint: use that \( \text{Aut}(\mathbb{P}^n) = \text{PGL}_n(\mathbb{C}) \) together with the fact that a birational map \( \phi : \mathbb{P}^n \to \mathbb{P}^n \) identifies line bundles on each side.

(4) Explain how working with complexes is really the same as working with graded modules over the algebra \( \mathbb{C}[x]/x^2 \). What are some of the key features of this algebra (e.g. Hopf structure). Explain how replacing this algebra with \( \mathbb{C}[x]/x^k \) leads us to the study of “Hopfological algebra” (recently explored by Khovanov et al with applications to categorification in mind).

(5) Discuss the functor \( f^! \) which is right adjoint to \( f_* \). This functor does not exist as a functor between abelian categories but only as a functor between derived categories (in particular, it is not the derived version of some functor). Illustrate what it has to be in some simple examples.

(6) Explain the idea of derived categories. What are quasi-isomorphisms and how do we localize. What is a triangulated category (focus on cones and distinguished triangles). Why do we want to go beyond the homotopy category to the derived category – one reason is that we want the exact sequence

\[
0 \to \mathbb{C}[x] \xrightarrow{z} \mathbb{C}[x] \to \mathbb{C}_0 \to 0
\]

of \( \mathbb{C}[x] \)-modules to be a distinguished triangle (explain). Explain how derived functors are necessary when working with derived categories (or how we are lead to derived functors if we decide we want to work with derived categories).

Discuss how the cone construction is an algebraic version of the geometric cone construction from topology. How does this give the homotopy category of spaces the structure of a triangulated category?
(7) What are ample line bundles? Discuss the relationship to very ample line bundles (e.g. Hartshorne Theorem II.7.6).

(8) Discuss the tor functor as the left derived functor associated to tensor product. Illustrate with some example computations. Explain how the projection formula can hold more generally (not just for locally free sheaves) if we take into account tors.

REFERENCES

[GNR] Eugene Gorsky, Andrei Negut and Jacob Rasmussen, Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology; arXiv:1608.07308