PROBLEMS: ALGEBRAIC GEOMETRY II

The following questions and problems came up naturally during class. They are not necessarily difficult but they should help understand the material from class.

(1) For a map $f : \mathcal{F} \to \mathcal{G}$ of sheaves show that $U \mapsto \ker(f(U))$ is a sheaf (and not just a presheaf).

(2) Define what it means to be a subsheaf.

(3) If $i : p \to X$ is the inclusion of a point show that $i_*\mathcal{O}_p$ is the skyscraper sheaf supported at $p \in X$ (here $\mathcal{O}_p$ is the structure sheaf of $p$).

(4) If $p \in X$ is a point then show that there exists a short exact sequence

$0 \to \mathcal{I}_p \xrightarrow{i} \mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_p \to 0$

where $\mathcal{I}_p$ is the ideal sheaf of $p$.

(5) Show that if $p, q \in \mathbb{P}^1$ are distinct points the natural map

$\mathcal{O}_{\mathbb{P}^1}(\pi_1, \pi_2) \to \mathcal{O}_p \oplus \mathcal{O}_q$

is surjective, where $\pi_1, \pi_2$ are the natural surjective maps as in (1).

(6) If $f : X \to Y$ and $\mathcal{G}$ is a sheaf on $Y$ describe the natural adjunction map

$\mathcal{G} \to f_*f^{-1}\mathcal{G}$.

(7) Show that the global section functor $\Gamma$ is left exact.

(8) What map of rings defines the inclusion of the point $(a_1, \ldots, a_n) \subset \mathbb{A}^n$.

(9) Describe the glueing maps for the standard cover by affines of $\mathbb{P}^n$.

(10) Show that a homogeneous polynomial $f(x_0, x_1)$ of degree $d$ carves out $d$ points in $\mathbb{P}^1$ (counted with multiplicity).

(11) Show that $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is an affine scheme. Show that $\mathbb{C}^2 \setminus \{(0, 0)\}$ is not an affine scheme.

(12) Find an example of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ so that for open $U \subset X$ the assignment

$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

defines a presheaf but not a sheaf.

(13) For a subscheme $i : Y \subset X$ show that $i^*i_*\mathcal{O}_Y \cong \mathcal{O}_Y$.

(14) If $i : Y \subset X = \text{Spec}(A)$ is a closed subscheme corresponding to an ideal $I \subset A$ describe $i_*\mathcal{O}_Y$ as $\widetilde{M}$ for an $A$-module $M$.

(15) For an open $j : U \subset X$ define $j_!(\mathcal{F})$ as the sheafification of

$V \mapsto \begin{cases} \mathcal{F}|_V & \text{if } V \subset U \\ 0 & \text{if } V \not\subset U \end{cases}$
Show that there exists a short exact sequence
\[ 0 \to j_!j^{-1} \mathcal{F} \to \mathcal{F} \to i_*i^{-1} \mathcal{F} \to 0 \]
where \( i : Y = X \setminus U \to X \).

(16) Explain why \( j_!(\mathcal{O}_U) \) is not a quasi-coherent sheaf in general.

(17) Is \( j_*\mathcal{O}_Y \) coherent where \( j : Y = \mathbb{C}^2 \setminus \{(0,0)\} \hookrightarrow \mathbb{C}^2 \)?

(18) Given an exact sequence
\[ 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \]
on an affine scheme \( X \) show that if \( \mathcal{F}_1 \) and \( \mathcal{F}_3 \) are quasi-coherent then so is \( \mathcal{F}_2 \).

Hint: you may use that \( \Gamma \) applied to (2) is exact if \( X \) is affine.

(19) Show that a morphism of line bundles \( \mathbb{C}_X \to L \) (where \( \mathbb{C}_X \) is the trivial line bundle on \( X \)) is equivalent to a section \( s : X \to L \).

(20) Show that a line bundle \( L \) on \( X \) is trivial if and only if there exists a nonvanishing section \( s : X \to L \) (i.e. section where \( s(x) \neq 0 \) for all \( x \in X \)).

(21) If \( L \) is a line bundle show that there exists another line bundle \( L^{-1} \) such that \( L \otimes L^{-1} \cong \mathbb{C}_X \).

(22) Show that for an \( \mathcal{O}_X \)-module \( \mathcal{F} \) we have
\[ \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(\mathcal{F}) \]
as vector spaces.

(25) If \( \Psi(V) \) denotes the coherent sheaf associated to a vector bundle \( V \) show that
\[ \Psi(V \otimes W) \cong \Psi(V) \otimes_{\mathcal{O}_X} \Psi(W). \]

(26) Show that for a vector bundle \( V \) on \( Y \) and morphism \( f : X \to Y \) we have
\[ f^*(\Psi(V)) \cong \Psi(V \otimes_Y X) \]
where \( V \otimes_Y X \) is the pullback (as a vector bundle) of \( V \) from \( Y \) to \( X \).

(28) Compute \( \dim \Gamma(\mathcal{O}_{\mathbb{P}^n}(m)) \) for \( m \geq 0 \).

(29) If \( X \) is proper and \( \mathcal{L} \) a line bundle on \( X \) show that both \( \Gamma(\mathcal{L}) \neq 0 \) and \( \Gamma(\mathcal{L}^{-1}) \neq 0 \) if and only if \( \mathcal{L} \) is trivial. Hint: just need to use that \( \Gamma(\mathcal{O}_X) \cong \mathbb{C} \).

(30) Describe the cokernel of the map \( x_k : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \) of sheaves on \( \mathbb{P}^n \).

(31) Check that if \( s \in \Gamma(\mathcal{L}) \) then \( s^n \in \Gamma(\mathcal{L}^{\otimes n}) \) where \( \mathcal{L} \) is a line bundle.

(32) Consider the map
\[ \phi : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}^r \]
given by \([(x_1, y_1), \ldots, (x_r, y_r)] \mapsto [f_0, \ldots, f_r]\) where

\[ f_i = \sum_{e_1 + \cdots + e_r = i} \prod_j x_j^{1-e_j} y_j^{e_j} \]

with \(0 \leq e_k \leq 1\). For example, if \(r = 2\) then \(\phi\) is given by

\[ \left([x_1, y_1], [x_2, y_2] \right) \mapsto [x_1 x_2, x_1 y_2 + x_2 y_1, y_1 y_2]. \]

(a) Show that \(\phi\) is well defined.

(b) Compute the number of points in \(\phi^{-1}(p)\) for a general point \(p \in \mathbb{P}^r\).

(38) If \(p, q \in \mathbb{P}^1\) are distinct points compute \(\Gamma(\mathcal{O}_{\mathbb{P}^1}(p - q))\).

(39) If \(H \subset \mathbb{P}^n\) is a hyperplane compute \(\dim(\mathcal{O}_{\mathbb{P}^n}(mH))\) for \(m \in \mathbb{Z}\).

(40) Show that \(PGL_{n+1}(\mathbb{C})\) acts faithfully on \(\mathbb{P}^n\).

(41) Show that \(Aut(\mathbb{A}^n)\) is given by the group of affine transformations on \(\mathbb{A}^n\). Hint: use that \(Aut(\mathbb{P}^n) = PGL_n(\mathbb{C})\) together with the fact that a birational map \(\phi : \mathbb{P}^n \to \mathbb{P}^n\) induces an isomorphism on \(\text{Pic}(\mathbb{P}^n)\).

(42) Consider a map \(B \to A\) of rings. This allows us to define

\[ f : A \otimes_B A \to A \quad (a, a') \mapsto aa'. \]

Check that \(d : A \to \ker(f)/\ker(f)^2\) given by \(a \mapsto a \otimes 1 - 1 \otimes a\) defines a \(B\)-derivation (i.e. \(d(aa') = ad(a') + a'd(a)\) and \(d(b) = 0\) for \(b \in B\)).

(43) Show that \(\det(\mathcal{F}^\vee) \cong \det(\mathcal{F})^{-1}\) for any locally free sheaf \(\mathcal{F}\).

(44) Show that if

\[ 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \]

is a short exact sequence of locally free sheaves then \(\det(\mathcal{F}_2) \cong \det(\mathcal{F}_1) \otimes \det(\mathcal{F}_3)\).

(45) Since \(\text{Pic}(\mathbb{P}_n) \cong \mathbb{Z}\) we know \(\omega_{\mathbb{P}_n} \cong \mathcal{O}_{\mathbb{P}_n}(\ell)\) for some \(\ell \in \mathbb{Z}\). Find \(\ell\).

(46) If \(X \subset Y\) are smooth varieties \((X \neq Y)\) show that

\[ \det(N_{X/Y}) \cong \omega_X \otimes \omega_Y^{-1}|_X. \]
Consider the commutative diagram

\[
\begin{array}{c}
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \\
\downarrow f \downarrow g \downarrow h \\
0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0
\end{array}
\]

where the rows are exact, \( f \) is injective and \( g \) is an isomorphism. Show that \( h \) is surjective and \( \text{coker}(f) \cong \ker(h) \).

Show from basic principles that \( \Omega_{X \times Y} \cong \pi_X^* \Omega_X \otimes_{\mathcal{O}_{X \times Y}} \pi_Y^* \Omega_Y \) where \( \pi_X \) and \( \pi_Y \) are the projections form \( X \times Y \) to \( X \) and \( Y \).

Suppose \( X \subset Y \) are smooth varieties and that \( X \) is the zero locus of a section of a vector bundle \( V \) whose rank is the codimension of \( X \) in \( Y \). Show that

\[ \omega_X \cong \omega_Y|_X \otimes_{\mathcal{O}_X} (\det V)|_X. \]

Find an example which illustrates why the homotopy category \( \text{Kom}(\mathcal{C}) \) of an abelian category \( \mathcal{C} \) is not necessarily abelian.

Given an example of an abelian category \( \mathcal{C} \) and an object \( A \in \mathcal{C} \) so that the functor \( \text{Hom}(A, -) \) is not exact.

Recall that Cartier divisors on an irreducible scheme \( X \) correspond to \( H^0(K_X^*/\mathcal{O}_X^*) \).

Use the short exact sequence

\[ 0 \rightarrow \mathcal{O}_X^* \rightarrow K_X^* \rightarrow K_X^*/\mathcal{O}_X^* \rightarrow 0 \]

to show that Cartier divisors on \( X \) modulo principal divisors are also parametrized by \( H^1(X, \mathcal{O}_X^*) \).

Give an example of a scheme \( X \) which is not affine but contains \( f_1, \ldots, f_n \in \Gamma(\mathcal{O}_X) \) so that \( U_i := \{ f_i \neq 0 \} \) are all open affine and \( X = \bigcup_i U_i \).

Calculate \( \check{H}^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) \) for all \( m \in \mathbb{Z} \) with respect to the standard cover \( U_0 \cup U_1 \) of \( \mathbb{P}^1 \).

Consider \( X = \mathbb{A}^2 \setminus (0, 0) \) where \( \mathbb{A}^2 = \text{Spec}(\mathbb{C}[x, y]) \). Using an appropriate cover of \( X \) show that

\[ \check{H}^1(X, \mathcal{O}_X) = \text{span}\{x^i y^j : i, j < 0\}. \]

In particular, this again shows that \( X \) is not affine.

Suppose \( X \) is projective and \( \mathcal{F} \) a coherent sheaf on \( X \). Define

\[ \chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}). \]

If \( 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0 \) is a short exact sequence show that \( \chi(\mathcal{F}_1) - \chi(\mathcal{F}_2) + \chi(\mathcal{F}_3) = 0 \).

Let \( E \subset \mathbb{P}^2 \) be a smooth curve carved out by a (homogeneous) degree 3 polynomial. Use the short exact sequence

\[ 0 \rightarrow I_E \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_E \rightarrow 0 \]
where $I_E$ is the ideal sheaf of $E$, to compute $H^i(E, \mathcal{O}_E)$ for $i \geq 0$. Hint: $I_E \cong \mathcal{O}_{\mathbb{P}^2}(-3)$.

(58) For $E$ as above, let $p \in E$ be an arbitrary point. Calculate $H^i(E, \mathcal{O}_E(np))$ for $n \in \mathbb{Z}, i \geq 0$. Hint: you can use the calculation of $H^i(E, \mathcal{O}_E)$ from the above exercise and induction on $n$.

(59) For $E$ as above, show that $\mathcal{O}_E(p)$ is ample but not very ample. Hint: first show that cohomology criterion for ampleness holds when the test sheaf $\mathcal{F}$ is locally free.

(60) Compute $\text{Ext}^i(\mathcal{O}_p, \mathcal{O}_p)$ where $p$ is a point in $\mathbb{P}^2$.

(61) Check that a closed inclusion $i : X \rightarrow Y$ is not a flat morphism ($X \neq Y$).

(62) Check that the flatness criterion from class implies that the fibres of a flat morphism over a smooth variety all have the same dimension.

(63) Consider the maps

\[
\mathbb{C}[t] \rightarrow \mathbb{C}[x, y]/xy \rightarrow \mathbb{C}[x]
\]

where the first map is given by $t \mapsto x + y$ and the second by $x \mapsto x, y \mapsto 0$. This induces maps

\[
X \xrightarrow{f} Y \xrightarrow{g} Z
\]

where $X = \text{Spec}\mathbb{C}[x], Y = \text{Spec}\mathbb{C}[x, y]/xy, Z = \text{Spec}\mathbb{C}[t]$. Determine which of $f, g, g \circ f$ are flat morphisms.

(64) Do the same for the maps

\[
\mathbb{C}[u, v] \cong \mathbb{C}[x^2, y^2] \rightarrow \mathbb{C}[x^2, xy, y^2] \rightarrow \mathbb{C}[x, y]
\]

where both maps are inclusions. Aside: what do these maps represent geometrically?