The following questions and problems came up naturally during class. They are not necessarily difficult but they should help understand the material from class.

(1) For a map $f : \mathcal{F} \to \mathcal{G}$ of sheaves show that $U \mapsto \ker(f(U))$ is a sheaf (and not just a presheaf).

(2) Write out what it means to be a subsheaf.

(3) If $i : p \to X$ is the inclusion of a point show that $i_* \mathcal{O}_p$ is the skyscraper sheaf supported at $p \in X$ (here $\mathcal{O}_p$ is the structure sheaf of $p$).

(4) If $p \in X$ is a point then show that there exists a short exact sequence

$$0 \to \mathcal{I}_p \xrightarrow{i^*} \mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_p \to 0$$

where $\mathcal{I}_p$ is the ideal sheaf of $p$.

(5) Show that if $p, q \in \mathbb{P}^1$ are distinct points the natural map

$$\mathcal{O}_{\mathbb{P}^1} \xrightarrow{\langle \pi_1, \pi_2 \rangle} \mathcal{O}_p \oplus \mathcal{O}_q$$

is surjective, where $\pi_1, \pi_2$ are the natural surjective maps as in (1).

(6) Show that the global section functor $\Gamma$ is left exact.

(7) If $f : X \to pt$ is the projection to a point show that $f_* = \Gamma$ where $\Gamma$ is the global sections functor.

(8) If $f : X \to Y$ and $\mathcal{G}$ is a sheaf on $Y$ describe the natural adjunction map

$$\mathcal{G} \to f_* f^{-1} \mathcal{G}.$$

(9) Given sheaves $\mathcal{F}, \mathcal{G}$ on $X$ show that

$$U \mapsto \text{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U)$$

defines a sheaf (usually denoted $\mathcal{H}om(\mathcal{F}, \mathcal{G})$).

(10) Recall the extension by zero functor $j!$: namely, for an open $j : U \hookrightarrow X$ we define $j_!(\mathcal{F})$ as the sheafification of

$$V \mapsto \begin{cases} \mathcal{F}|_V & \text{if } V \subset U \\ 0 & \text{if } V \not\subset U \end{cases}$$

Show that there exists a short exact sequence

$$0 \to j_! j^{-1} \mathcal{F} \to \mathcal{F} \to i_* i^{-1} \mathcal{F} \to 0.$$

where $i : Y = X \setminus U \to X$. Note that $j^{-1} \mathcal{F}$ is just the restriction $\mathcal{F}|_U$ while the second (nontrivial) map above is the adjunction morphism.
(11) Show that a homogeneous polynomial \( f(x_0, x_1) \) of degree \( d \) carves out \( d \) points in \( \mathbb{P}^1 \) (counted with multiplicity).

(12) Show that \( \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \) is an affine scheme.

(13) Show that \( X = \mathbb{C}^2 \setminus \{(0,0)\} \) is not an affine scheme. [You may use that \( X \) is not isomorphic to \( \mathbb{C}^2 \) topologically – which is clear since \( \mathbb{C}^2 \) is contractible whereas \( X \) is not.]

(14) What map of rings defines the inclusion of the point \((a_1, \ldots, a_n) \hookrightarrow \mathbb{A}^n\).

(15) Describe the glueing maps for the standard cover (discussed in class) by affines of \( \mathbb{P}^n \) [Hint: Recall the case \( n = 1 \) worked out as an example. In that case we had two copies of \( \mathbb{C} \) which are glued along \( \mathbb{C}^\times \) by the map \( z \mapsto z^{-1} \).]

(16) Identify the complement of the natural embedding \( \{x_n = 0\} = \mathbb{P}^{n-1} \subset \mathbb{P}^n \).

(17) Find an example of \( \mathcal{O}_X \)-modules \( \mathcal{F} \) and \( \mathcal{G} \) so that for open \( U \subset X \) the assignment
\[
U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)
\]
defines a presheaf but not a sheaf.

(18) Show that for an inclusion \( i : Y \hookrightarrow X \) we have \( i^* i_* \mathcal{O}_Y \cong \mathcal{O}_Y \).

(19) If \( i : Y \hookrightarrow X = \text{Spec}(A) \) is a closed subscheme corresponding to an ideal \( I \subset A \) describe \( i_* \mathcal{O}_Y \) as \( \tilde{M} \) for an \( A \)-module \( M \).

(20) Is \( j_* \mathcal{O}_Y \) coherent where \( j : Y = \mathbb{C}^2 \setminus \{(0,0)\} \hookrightarrow \mathbb{C}^2 \)?

(21) Using the notation from (2), explain why \( j_!(\mathcal{O}_U) \) is not a quasi-coherent sheaf (in general).

(22) Consider an exact sequence of sheaves
\[
0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0
\]
on a scheme \( X \). Show that if \( \mathcal{F}_1 \) and \( \mathcal{F}_3 \) are quasi-coherent then so is \( \mathcal{F}_2 \). Hint: This is a local question. You can then use that \( \Gamma \) is exact to obtain an exact sequence of the form
\[
0 \rightarrow \mathcal{F}_1 \rightarrow \Gamma(\mathcal{F}_2) \rightarrow \mathcal{F}_3 \rightarrow 0.
\]

(23) Show that if \( U, V \subset X \) are open affine subschemes then \( U \cap V \) is also affine. Hint: consider the following fiber diagram
\[
\begin{array}{ccc}
U \cap V & \longrightarrow & U \times V \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta} & X \times X
\end{array}
\]
where the vertical maps are inclusions and \( \Delta \) is the diagonal embedding. You may then use that a closed subscheme of an affine scheme is also affine.

(24) Find an example of a coherent sheaf \( \mathcal{F} \) on \( X \) such that \( f_*(\mathcal{F}) \) is not coherent (just quasi-coherent). Hint: think affine.

(25) Show that for an \( \mathcal{O}_X \)-module \( \mathcal{F} \) we have
\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(\mathcal{F}).
\]
(26) Suppose that \( i : X \to Y \) is a closed immersion and suppose \( F \) is a coherent \( \mathcal{O}_X \)-module. Show that \( i_* (F) \) is a coherent \( \mathcal{O}_Y \)-module (apriori it is just quasi-coherent).

(27) Show that a morphism of line bundles \( \mathbb{C}_X \to L \) (where \( \mathbb{C}_X \) is the trivial line bundle on \( X \)) is equivalent to a section \( s : X \to L \).

(28) Show that a line bundle \( L \) on \( X \) is trivial if and only if there exists a nonvanishing section \( s : X \to L \) (i.e. section where \( s(x) \neq 0 \) for all \( x \in X \)).

(29) If \( L \) is a line bundle show that there exists another line bundle \( L^{-1} \) such that \( L \otimes L^{-1} \cong \mathbb{C}_X \).

(30) Show that for an \( \mathcal{O}_X \)-module \( F \) we have 

\[
\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F) \cong \Gamma(F)
\]

as vector spaces.

(31) If \( \Psi(V) \) denotes the coherent sheaf associated to a vector bundle \( V \) show that 

\[
\Psi(V \otimes W) \cong \Psi(V) \otimes_{\mathcal{O}_X} \Psi(W).
\]

(32) Show that for a vector bundle \( V \) on \( Y \) and morphism \( f : X \to Y \) we have 

\[
f^*(\Psi(V)) \cong \Psi(V \otimes_Y X)
\]

where \( V \otimes_Y X \) is the pullback (as a vector bundle) of \( V \) from \( Y \) to \( X \).

(33) Show that \( \Gamma(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0 \).

(34) Compute \( \dim \Gamma(\mathcal{O}_{\mathbb{P}^n}(m)) \) for \( m \geq 0 \).

(35) If \( X \) is proper and \( \mathcal{L} \) a line bundle on \( X \) show that both \( \Gamma(\mathcal{L}) \neq 0 \) and \( \Gamma(\mathcal{L}^{-1}) \neq 0 \) if and only if \( \mathcal{L} \) is trivial. Hint: just need to use that \( \Gamma(\mathcal{O}_X) \cong \mathbb{C} \).

(36) Show that if 

\[
0 \to F_1 \to F_2 \to F_3 \to 0
\]

is a short exact sequence of locally free sheaves then \( \det(F_3) \cong \det(F_1) \otimes \det(F_3) \).

(37) Describe the cokernel of the map \( x_k : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \) of sheaves on \( \mathbb{P}^n \).

(38) Check that if \( s \in \Gamma(\mathcal{L}) \) then \( s^n \in \Gamma(\mathcal{L} \otimes \mathcal{L}^{-1}) \) where \( \mathcal{L} \) is a line bundle.

(39) Consider the map 

\[
\phi : \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \to \mathbb{P}^r
\]

given by \( ([x_1, y_1], \ldots, [x_r, y_r]) \mapsto [f_0, \ldots, f_r] \) where 

\[
f_i = \sum_{e_1 + \cdots + e_r = i} \prod_j x_j^{1-e_j} y_j^{e_j}
\]

with \( 0 \leq e_k \leq 1 \). For example, if \( r = 2 \) then \( \phi \) is given by 

\[
([x_1, y_1], [x_2, y_2]) \mapsto [x_1 x_2, x_1 y_2 + x_2 y_1, y_1 y_2].
\]

(a) Show that \( \phi \) is well defined. 

(b) Compute the number of points in \( \phi^{-1}(p) \) for a general point \( p \in \mathbb{P}^r \).
(c) (Harder) Show that
\[ \phi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \bigotimes_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(1). \]

(40) Show that a line bundle \( \mathcal{L} \) on \( X \) is generated by global section \( s_1, \ldots, s_n \in \Gamma(\mathcal{L}) \) if for any point \( p \in X \) there exists some \( s_i \) with \( s_i(p) \neq 0 \).

(41) Consider a morphism \( f : X \to Y \) of schemes and a line bundle \( \mathcal{L} \) on \( Y \). If sections \( \{ s_i \} \) generate \( \mathcal{L} \) show that \( \{ f^*s_i \} \) generate \( f^*\mathcal{L} \) on \( X \).

(42) Show that \( PGL_{n+1}(\mathbb{C}) \) acts faithfully on \( \mathbb{P}^n \).

(43) Show that all line bundles on \( \mathbb{P}^n \) are of the form \( \mathcal{O}_{\mathbb{P}^n}(\ell) \) for some \( \ell \in \mathbb{Z} \). [Hint: this requires thinking about the vanishing loci of sections.]

(44) Show that \( Aut(\mathbb{A}^n) \) is given by the group of affine transformations on \( \mathbb{A}^n \). Hint: use that \( Aut(\mathbb{P}^n) = PGL_n(\mathbb{C}) \) together with the fact that a birational map \( \phi : \mathbb{P}^n \to \mathbb{P}^n \) identifies line bundles on each side. This can be turned/expanded into a possible project.

(45) Consider a map \( B \to A \) of rings. This allows us to define
\[ f : A \otimes_B A \to A \quad (a,a') \mapsto aa'. \]
Check that \( d : A \to \ker(f)/\ker(f)^2 \) given by \( a \mapsto a \otimes 1 - 1 \otimes a \) defines a \( B \)-derivation (i.e. \( d(aa') = ad(a') + a'd(a) \) and \( d(b) = 0 \) for \( b \in B \)).

(46) Show that \( \det(\mathcal{F}^\vee) \cong \det(\mathcal{F})^{-1} \) for any locally free sheaf \( \mathcal{F} \).

(47) For \( n \geq 1 \) we have \( \omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(1) \) for some \( \ell \in \mathbb{Z} \). Find \( \ell \).

(48) If \( X \subset Y \) are smooth varieties \( (X \neq Y) \) show that
\[ \det(N_{X/Y}) = \omega_X \otimes \omega_Y^{-1}|_X. \]

(49) Consider a commutative diagram
\[
\begin{array}{ccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & & & \\
0 & \to & A' & \to & B' & \to & C' & \to & 0
\end{array}
\]
where the rows are exact, \( f \) is injective and \( g \) is an isomorphism. Show that \( h \) is surjective and that \( \text{coker}(f) \cong \text{ker}(h) \).

(50) Show from basic principles that \( \Omega_{X \times Y} \cong \pi_X^*\Omega_X \otimes_{\mathcal{O}_{X \times Y}} \pi_Y^*\Omega_Y \) where \( \pi_X \) and \( \pi_Y \) are the projections from \( X \times Y \) to \( X \) and \( Y \).

(51) Suppose \( X \subset Y \) are smooth varieties and that \( X \) is the zero locus of a section of a vector bundle \( \mathcal{V} \) whose rank is the codimension of \( X \) in \( Y \). Show that
\[ \omega_X \cong \omega_Y|_X \otimes (\text{det} \mathcal{V})|_X. \]

(52) Suppose \( C \subset \mathbb{P}^2 \) is a smooth curve of degree \( d \). Find an \( \ell \in \mathbb{Z} \) and show that \( \omega_C \cong i^*(\mathcal{O}_{\mathbb{P}^2}(\ell)) \) where \( i : C \to \mathbb{P}^2 \) denotes the embedding.

(53) Find an example which illustrates why the homotopy category \( Kom(\mathcal{C}) \) of an abelian category \( \mathcal{C} \) is not necessarily abelian.
(54) Given an example of an abelian category $\mathcal{C}$ and an object $A \in \mathcal{C}$ so that the functor $\text{Hom}(A, -)$ is not exact.

(55) Consider an exact sequence of sheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$.

a) If $\mathcal{F}_1$ is flasque show that we have an exact sequence

$$0 \to \Gamma(\mathcal{F}_1) \to \Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3) \to 0.$$ 

b) If $\mathcal{F}_1$ and $\mathcal{F}_2$ are both flasque show that $\mathcal{F}_3$ is also flasque.

(56) Give an example of a scheme $X$ which is not affine but contains $f_1, \ldots, f_n \in \Gamma(\mathcal{O}_X)$ so that $U_i := \{ f_i \neq 0 \}$ are all open affine and $X = \cup_i U_i$.

(57) Calculate $\check{H}^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ for all $m \in \mathbb{Z}$ with respect to the standard cover $U_0 \cup U_1$ of $\mathbb{P}^1$.

(58) Show that $\check{H}^0(X, \mathcal{F}) \cong \Gamma(\mathcal{F})$ using any cover of $X$.

(59) If $\iota : U \to X$ is an open embedding and $\mathcal{F}$ a sheaf on $X$ then the natural map $\iota_* : \mathcal{F} \to \iota_* (\mathcal{F}|_U)$ becomes an isomorphism when restricted back to $U$.

(60) Consider $X = \mathbb{A}^2 \setminus (0, 0)$ where $\mathbb{A}^2 = \text{Spec} \mathbb{C}[x, y]$. Using an appropriate cover of $X$ show that

$$\check{H}^1(X, \mathcal{O}_X) = \text{span}\{ x^i y^j : i, j < 0 \}.$$ 

In particular, this again shows that $X$ is not affine.

(61) $X = \mathbb{A}^2 \setminus \{(0, 0)\}$ and consider the embedding $\iota : X \to \mathbb{A}^2$. Show that $\iota_*$ is not exact. Hint: consider the following exact sequence on $X$

$$0 \to \mathcal{O}_X \xrightarrow{(x, y)} \mathcal{O}_{\mathbb{P}^2}(-y, x) \xrightarrow{0} \mathcal{O}_X \to 0.$$

(62) Suppose $X$ is projective and $\mathcal{F}$ a coherent sheaf on $X$. Define the Euler characteristic as

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}).$$

If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence show that $\chi(\mathcal{F}_1) - \chi(\mathcal{F}_2) + \chi(\mathcal{F}_3) = 0$.

(63) For a projective variety $X$ and a coherent sheaf $\mathcal{F}$ on it show that $\chi(\mathcal{F}(n))$ is a polynomial in $n$. Hint: one idea is to use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r}(n) \to \mathcal{O}_{\mathbb{P}^r}(n + 1) \to \mathcal{O}_{\mathbb{P}^{r+1}}(n + 1) \to 0$$

tensor by $\mathcal{F}$ and then use induction on the dimension but this is a bit tricky because tensoring is not exact. One solution is to consider the Euler characteristic of more general complexes $\mathcal{F}$ (i.e. objects in the derived category) but this requires being a bit more comfortable with derived tensor products.

(64) Let $E \subset \mathbb{P}^2$ be a smooth curve of degree 3. Use the exact sequence

$$0 \to \mathcal{I}_E \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_E \to 0$$

to compute $H^i(E, \mathcal{O}_E)$ for $i \geq 0$. Hint: $\mathcal{I}_E \cong \mathcal{O}_{\mathbb{P}^2}(-3)$. 


(65) For $E$ as above and $p \in E$ denote by $\mathcal{I}_p$ the ideal sheaf of $p$ (which is a line bundle). Compute $H^i(E, \mathcal{I}_p^\otimes n)$ for $n \in \mathbb{Z}$ and $i \geq 0$. Hint: use the computation of $H^i(E, \mathcal{O}_E)$ above and induction on $n$.

(66) For $E$ as above show that $\mathcal{I}_p \not\cong \mathcal{I}_q$ if $p \neq q \in E$. Hint: Show that $H^0(E, \mathcal{I}_p \otimes \mathcal{I}_q') = 0$ if $p \neq q$.

(67) For $E$ as above let $p \in E$ be some point and compute $H^i(E, \mathcal{O}_E(np))$ for $n \in \mathbb{Z}$ and $i \geq 0$. Hint: use the computation of $H^i(E, \mathcal{O}_E)$ above and induction on $n$.

(68) Compute $\mathcal{E}xt^i(\mathcal{O}_p, \mathcal{O}_p)$ for $p = [0, 0, 1] \in \mathbb{P}^2$. Hint: one has an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_p \to 0$$

corresponding to the exact sequence of $\mathbb{C}[x, y]$-modules

$$0 \to \mathbb{C}[x, y] \xrightarrow{(x, y)} \mathbb{C}[x, y]^{\oplus 2} \xrightarrow{(y, -x)} \mathbb{C}[x, y] \to 0.$$

(69) Consider the origin $p \in \mathbb{A}^2 = \text{Spec} \mathbb{C}[x, y]$. Describe $\{p\} \in \mathbb{A}^2$ as carved out by a section of a vector bundle and check that the resulting Koszul resolution of $\mathcal{O}_p$ is indeed exact. Hint: this is a complex we have seen before.