Consider the regular Sturm-Liouville differential equation
\[ \frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi + \lambda \sigma(x)\phi = 0. \]
where \( \phi = \phi(x) \) is a smooth function on the interval \([a, b]\).

1. [5 points] Briefly explain why the Sturm-Liouville differential equation is relevant either to the study of heat flow on a rod or the study of vibrating strings.

One can obtain the general solution to the heat flow (or vibrating string) problem by using separation of variables. When you separate the space variable \( x \) from the time variable \( t \) you obtain two ordinary differential equations. The ODE involving \( t \) will be easy to solve but in the more general version of the heat flow PDE the resulting ODE involving \( x \) will be a Sturm-Liouville differential equation which we need to understand.

2. [5 points] Consider the space of smooth functions \( f(x) \) on \([a, b]\) such that \( f(a) = 0 \) and \( f(b) = 1 \). Explain why this is not a vector space.

Consider any smooth function \( f(x) \) on \([a, b]\) satisfying \( f(a) = 0 \) and \( f(b) = 1 \) (for instance, the linear function \( \frac{1}{b-a}(x-a) \)). Then for any constant \( c \neq 1 \) the function \( cf(x) \) will no longer be equal to 1 when evaluated at \( x = b \). Thus \( f(x) \) is in this space but \( cf(x) \) is not in this space which violates one of the axioms of being a vector space.

Even more to the point, the function \( f(x) = 0 \) is not in this space and this already shows it is not a vector space because any vector space must contain the “zero” vector.

3. [15 points] Let \( V \) be the infinite dimensional vector space of smooth functions \( f(x) \) on \([a, b]\) such that \( f(a) = 0 \) and \( f(b) = 0 \). Show that \( \langle f(x), g(x) \rangle = \int_a^b f(x)g(x)\sigma(x)dx \) defines an inner product on \( V \).

We need to show that for any three smooth functions \( f(x), g(x) \) and \( h(x) \) on \([a, b]\) which are zero at \( a \) and \( b \) we have

- \( \langle cf + g, h \rangle = c\langle f, h \rangle + \langle g, h \rangle \) where \( c \) is any constant
- \( \langle f, g \rangle = \langle g, f \rangle \)
- \( \langle f, f \rangle > 0 \) unless \( f(x) = 0 \)

The first fact is a consequence of the linearity of the integral, namely
\[
\langle cf + g, h \rangle = \int_a^b (cf + g)\sigma dx = c\int_a^b f\sigma dx + \int_a^b gh\sigma dx = c\langle f, h \rangle + \langle g, h \rangle
\]

The second fact is obvious since \( \int_a^b f\sigma dx = \int_a^b g\sigma dx \). The third fact uses that \( \sigma > 0 \) to conclude that \( f^2\sigma \geq 0 \). If \( f \) is not identically zero then it’s nonzero at some point \( x = x_0 \). So in a small interval \( I \) containing \( x_0 \) it
4. [35 points] Let $L$ be the linear differential operator on $V$ defined by

$$L(\phi) = \frac{1}{\sigma(x)} \left( \frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi \right).$$

Show that $L$ is self-adjoint with respect to the inner product $\langle f, g \rangle = \int_a^b f g \sigma dx$. [Hint: use integration by parts like in the proof of Lagrange’s identity].

Let $T$ be the linear operator

$$T(\phi) = \sigma(x)L(\phi) = \left( \frac{d}{dx} \left( p(x) \frac{d\phi}{dx} \right) + q(x)\phi \right).$$

Lagrange’s identity states that for any smooth functions $f, g$

$$\int_a^b T(f) \cdot g dx = \int_a^b f \cdot T(g) dx$$

Replacing $T(f)$ by $L(f)\sigma(x)$ and $T(g)$ by $L(g)\sigma(x)$ gives

$$\int_a^b L(f)\sigma(x)g dx = \int_a^b f L(g)\sigma(x) dx$$

which proves that $L$ is self-adjoint with respect to the inner product $\langle f, g \rangle = \int_a^b f g \sigma dx$.

5. [20 points] Assuming $L$ as defined above is self-adjoint show that two eigenfunctions $\phi_1$ and $\phi_2$ of $L$ corresponding to different eigenvalues $\lambda_1$ and $\lambda_2$ are orthogonal with respect to the inner product $\langle f, g \rangle = \int_a^b f g \sigma dx$.

By the hypothesis we know $L(\phi_1) = \lambda_1 \phi_1$ and $L(\phi_2) = \lambda_2 \phi_2$. Also, since $L$ is self-adjoint we know $\langle L(\phi_1), \phi_2 \rangle = \langle \phi_1, L(\phi_2) \rangle$. Substituting we get

$$\langle \lambda_1 \phi_1, \phi_2 \rangle = \langle \phi_1, \lambda_2 \phi_2 \rangle$$

from which it follows

$$\lambda_1 \langle \phi_1, \phi_2 \rangle = \lambda_2 \langle \phi_1, \phi_2 \rangle.$$

Since $\lambda_1 \neq \lambda_2$ this means $\langle \phi_1, \phi_2 \rangle = 0$.

6. [15 points] Denote the eigenfunctions of $L$ by $\phi_1(x), \phi_2(x), \ldots$ and let $f(x)$ be a smooth function on $[a, b]$. Determine the generalized Fourier series $f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$ of $f(x)$ (ie. determine the generalized Fourier coefficients $a_n$).

We know that $\langle \phi_i(x), \phi_j(x) \rangle = 0$ if $i \neq j$ since any two eigenfunctions corresponding to different eigenvalues are orthogonal. Thus, if we take the inner product of $\sum_{n=1}^{\infty} a_n \phi_n(x)$ with $\phi_i$ we get

$$\langle \sum_{n=1}^{\infty} a_n \phi_n(x), \phi_i(x) \rangle = \sum_{n=1}^{\infty} a_n \langle \phi_n, \phi_i \rangle = a_i \langle \phi_i, \phi_i \rangle$$

Thus $\langle f, \phi_i \rangle = a_i \langle \phi_i, \phi_i \rangle$ from which we get that the generalized Fourier coefficient $a_i$ is given by

$$a_i = \frac{\langle f, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}.$$
7. [5 points] If \( f(x) \) is smooth on \([a, b]\) is it equal to its generalized Fourier series from the previous question? Give an example of a function on \([a, b]\) (not necessarily continuous) whose generalized Fourier series is *not* equal to it (explain why not).

Let \( x = x_0 \in (a, b) \) be any point and \( f \) any function discontinuous at \( x_0 \). Then the generalized Fourier series at \( x = x_0 \) converges to

\[
y_0 = \frac{1}{2} \left( \lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x) \right)
\]

so that if we define \( f(x_0) \) to be anything other than \( y_0 \) then the generalized Fourier series and the function \( f(x) \) will be different. Along similar lines, you could also choose a function on \([a, b]\) whose periodic extension is discontinuous at \( a \) or \( b \) (in a way this is the same example as above where you choose \( x_0 \) to be either \( a \) or \( b \)).