Math 381 Fall 2006 Final Exam
Instructor: Sabin Cautis
Take home: due back Friday, December 15, 2006

Instructions: This is a take home exam. You may use only your textbook and notes for help. You have three hours. Do all the problems. You must show your work to receive full credit on a problem. Use separate, stapled sheets to do so.
Please print your name clearly here.

Print name: ________________________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam apart from help from my textbook and notes.

1a. ________/15

1b. ________/5

1c. ________/10

2. ________/15

3a. ________/5

3b. ________/5

4a. ________/15

4b. ________/15

5a. ________/10

5b. ________/10

6a. ________/15

6b. ________/10

7. ________/35

8. ________/35
1. (a) [15 points] Give the Fourier series of $e^{-|x|}$ on the interval $-L \leq x \leq L$. Note: you can leave the expressions for the Fourier coefficients as integrals (ie you don’t need to evaluate them).
(b) [5 points] Explain why the answer in part (a) is a cosine Fourier series (ie there are only cosine terms).
(c) [10 points] Sketch the Fourier series of $e^{-|x|}$ from part (a) for $-\infty < x < \infty$.

2. [15 points] Let $u(x, y, z)$ be a solution to the Laplace equation
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.
\]
on some domain $D$ of $\mathbb{R}^3$. Amazingly, such a solution always satisfies the mean value theorem – namely, the mean value of $u$ on any sphere centered at a point $p$ is equal to the value of $u$ at the point $p$. Explain why this means that if $u$ is constant on the boundary of $D$ then it is constant everywhere. [Hint: if $u$ is not constant can its maximal/minimal value occur in the interior of $D$?]

3. (a) [5 points] If $L : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear operator \[
\begin{pmatrix} 3 & -1 \\ 0 & 4 \end{pmatrix}
\]
what is the adjoint operator $L^*$ of $L$ (with respect to the standard dot product)?
(b) [5 points] Is $L$ from part (a) self-adjoint? Why?

4. (a) [15 points] Let $L$ be a self-adjoint linear operator on a vector space $V$ equipped with an inner product $\langle \cdot, \cdot \rangle$. If $L(\phi_1) = \lambda_1 \phi_1$ and $L(\phi_2) = \lambda_2 \phi_2$ where $\lambda_1 \neq \lambda_2$ (ie we have two eigenvectors with distinct eigenvalues) show that $\langle \phi_1, \phi_2 \rangle = 0$ (ie the eigenvectors are orthogonal).
(b) [15 points] An example of a self-adjoint linear operator is the Sturm-Liouville operator where the inner product is $\langle f, g \rangle = \int_a^b f(x)g(x)dx$. Let $\phi_1, \phi_2, \ldots$ denote the eigenfunctions of the Sturm-Liouville PDE which, as we learned, have distinct eigenvalues. Show that if $f = \sum_{n=1}^{\infty} a_n \phi_n$ and $g = \sum_{n=1}^{\infty} b_n \phi_n$ then $\langle f, g \rangle = \sum_{n=1}^{\infty} a_n b_n \langle \phi_n, \phi_n \rangle$ (you can assume the operations of summation and integration can be interchanged).

5. (a) [10 points] Explain why (for a fixed $m$) the associated Legendre functions $P_m^l(x)$ are orthogonal with respect to $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$.
(b) [10 points] Suppose $f(x)$ is a smooth function on $[-1, 1]$ and $f(x) = \sum_{n=1}^{\infty} a_n P_m^l(x)$ show that $a_n = \int_{-1}^{1} f(x) P_m^l(x) dx / \int_{-1}^{1} P_m^l(x) P_m^l(x) dx$.

6. (a) [15 points] Show that the Fourier transform $\mathcal{F}$ is a linear operator – namely, show that $\mathcal{F}(c_1 f(x) + c_2 g(x)) = c_1 \mathcal{F}(f(x)) + c_2 \mathcal{F}(g(x))$.
(b) [10 points] Is it true that $\mathcal{F}(f(x)g(x)) = \mathcal{F}(f(x))\mathcal{F}(g(x))$? Explain.

7. [35 points] Compute (from the definition) the Fourier transform of $e^{-|x|}$.

8. [35 points] Suppose $u(x, t)$ is a solution to the 1-dimensional wave PDE
\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]
where $-\infty < x < \infty$. Assuming $u(x, t) \to 0$ as $x \to \pm \infty$ find the general solution for $\mathcal{F}(u)$. [Hint: first apply the FT to the PDE]