1. section 3.1, #4
We have \( \frac{\partial f}{\partial x} = -y^2e^{-xy^2} + 4x^3y^3 \) and \( \frac{\partial f}{\partial y} = -2xye^{-xy^2} + 3x^4y^2 \).
Hence
\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= y^4e^{-xy^2} + 12x^2y^3 \\
\frac{\partial^2 f}{\partial x \partial y} &= -2ye^{-xy^2} + 2xy^3e^{-xy^2} + 12x^3y^2 \\
\frac{\partial^2 f}{\partial y \partial x} &= -2ye^{-xy^2} + 2xy^3e^{-xy^2} + 12x^3y^2 \\
\frac{\partial^2 f}{\partial y^2} &= -2xe^{-xy^2} + 4x^2y^2e^{-xy^2} + 6x^4y
\end{align*}
\]
Notice that \( \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \) which verifies Theorem 1.

2. section 3.1, #22
a) \( g_t = -e^{-t} \sin x \) while \( g_x = e^{-t} \cos x \) and \( g_{xx} = -e^{-t} \sin x \) which shows \( g_t = g_{xx} \).

b) The slice \( t = 0 \) is the graph of \( \sin x \) shifted up by 2. The slice \( t = 1 \) you get the graph of \( e^{-1} \sin x \) shifted up by 2. In general slices \( t = \text{constant} \) will look like some multiple of \( \sin x \) shifted up by 2 with this multiple (the amplitude) tending to 0 as \( t \to \infty \).

c) \( g(x,t) \to 2 \) as \( t \to \infty \). This corresponds to the fact that the heat distributes itself evenly along the rod if we wait long enough.

3. section 3.2, #2
\( f(0,0) = 1 \). Now \( \frac{\partial f}{\partial x} = -2x/(x^2 + y^2 + 1)^2 \) and \( \frac{\partial f}{\partial y} = -2y/(x^2 + y^2 + 1)^2 \) so that evaluated at \( (0,0) \) these are both zero.
Next
\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= -2/(x^2 + y^2 + 1) + 8x^2/(x^2 + y^2 + 1)^3 \\
\frac{\partial^2 f}{\partial x \partial y} &= 4xy/(x^2 + y^2 + 1)^3 \\
\frac{\partial^2 f}{\partial y^2} &= -2/(x^2 + y^2 + 1)^2 + 8y^2/(x^2 + y^2 + 1)^3
\end{align*}
\]
which, when evaluated at \( (0,0) \) give -2, 0, -2.
Thus the 2nd order Taylor expansion is
\[
0 + (0,0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + 1/2(x,y) \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -x^2 - y^2
\]

4. section 3.2, #6
\( f(1,0) = 1 \). Now \( \frac{\partial f}{\partial x} = 2(x-1)e^{(x-1)^2} \cos y \) and \( \frac{\partial f}{\partial y} = -e^{(x-1)^2} \sin y \) so that evaluated at \( (1,0) \) these are both 0.
Next
\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 2e^{(x-1)^2} \cos y + 4(x-1)^2e^{(x-1)^2} \cos y
\end{align*}
\]
\[
\frac{\partial^2 f}{\partial x \partial y} = -2(x-1)e^{(x-1)^2} \sin y \\
\frac{\partial^2 f}{\partial y^2} = -e^{(x-1)^2} \cos y
\]

which, when evaluated at \((1,0)\) give 2, 0, -1.

Thus the 2nd order Taylor expansion is

\[
\frac{1}{2}(x-1, y) \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} = (x-1)^2 - y^2/2
\]

5. section 3.2, #7b

Let’s take some derivatives of \(h(x) = e^{-1/x}\).

We have

\[
h'(x) = \frac{1}{x^2}e^{-1/x} \\
h''(x) = -\frac{2}{x^3}e^{-1/x} + \frac{1}{x^4}e^{-1/x}
\]

If we repeat it’s clear we are going to get more and more terms of the form \(a/x^n e^{-1/x}\). However, for any \(n\) the limit of \(1/x^n e^{-1/x}\) as \(x\) tends to zero is zero. This means that

\[
\lim_{x \to 0^+} h^{(n)}(x) = 0
\]

for all \(n\) which shows that \(f(x)\) is differentiable to arbitrary order (this is often called infinitely differentiable or \(C^{\infty}\)).

On the other hand, since all the derivatives of \(f\) evaluated at zero are zero the Taylor series is identically zero (and hence does not equal \(f\)). This means that \(f\) is not analytic.

Essentially, what is happening is that the Taylor series only sees the left part of the function \(f\) (where it is identically zero). In most examples we deal with the Taylor series equals the function (i.e. the function is analytic) so the function \(f\) in this exercise is a good counter-example to keep in mind.