1. section 2.5, #4

\[ h(x, y) = \frac{e^{-2x-2y} + e^{2xy}}{e^{-2x-2y} - e^{2xy}} \]

so

\[ \frac{\partial h}{\partial x} = \frac{\left( -2e^{-2x-2y} + 2ye^{2xy} \right) \left( e^{-2x-2y} - e^{2xy} \right) - \left( e^{-2x-2y} + e^{2xy} \right) \left( -2e^{-2x-2y} - 2ye^{2xy} \right)}{(e^{-2x-2y} - e^{2xy})^2} \]

\[ = \frac{4e^{-2x-2y} + 2ye^{-2x-2y} + 2xy}{(e^{-2x-2y} - e^{2xy})^2} \]

where the second line follows from the first after some simplifying.

On the other hand,

\[ \frac{\partial f}{\partial u} = 2u(u^2 - v^2) - (u^2 + v^2)(2u) = \frac{-4uv^2}{(u^2 - v^2)^2} \]

and

\[ \frac{\partial f}{\partial v} = 2v(u^2 - v^2) - (u^2 + v^2)(-2v) = \frac{4u^2v}{(u^2 - v^2)^2} \]

Also \( \frac{\partial u}{\partial x} = -e^{-x-y} \) and \( \frac{\partial v}{\partial x} = ye^{xy} \). Hence

\[ \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = \frac{4e^{-2x-2y+2xy}}{(e^{-2x-2y} - e^{2xy})^2}(-e^{-x-y}) + \frac{4e^{-2x-2y+xy}}{(e^{-2x-2y} - e^{2xy})^2}(ye^{xy}) \]

which upon closer inspection is the same as what we calculated above for \( \frac{\partial h}{\partial x} \).

2. section 2.5, #8

We use the chain rule to get:

\[ \frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \rho} \]

Now \( \frac{\partial x}{\partial \rho} = \cos \theta \sin \phi \) and \( \frac{\partial y}{\partial \rho} = \sin \theta \sin \phi \) while \( \frac{\partial z}{\partial \rho} = \cos \phi \). Thus we get

\[ \frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x} (\cos \theta \sin \phi) + \frac{\partial f}{\partial y} (\sin \theta \sin \phi) + \frac{\partial f}{\partial z} (\cos \phi) \]

The other two partials are computed similarly.

3. section 2.5, #10
We have
\[ D(f \circ g)(0,0) = Df(g(0,0))Dg(0,0) \]
\[ = \begin{pmatrix} e^{u-w} \\ -\sin(v+u) + \cos(u+v+w) & 0 \\ -\sin(v+u) + \cos(u+v+w) & \cos(u+v+w) \end{pmatrix} \begin{pmatrix} 0 \\ -e^{u-w} \\ -\sin(v+u) + \cos(u+v+w) & \cos(u+v+w) \end{pmatrix} \]
\[ = \begin{pmatrix} 1 \\ -\sin(2) + \cos(3) & 0 \\ -\sin(2) + \cos(3) & \cos(3) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \]
\[ = \begin{pmatrix} 1 \\ -\sin(2) + \cos(3) & 1 \\ -\sin(2) + \cos(3) & -\cos(3) \end{pmatrix} \]

4. section 2.5, #15
We want to compute the tangent vector to \( f(c(t)) \) at \( f(c(0)) = f(0,0) = (1,1) \). By the chain rule this is
\[ Df(c(0,0))c'(0,0) = \begin{pmatrix} e^{x+y} \\ e^{x+y} \\ -e^{x-y} \end{pmatrix} \begin{pmatrix} c'_1(t) \\ c'_2(t) \end{pmatrix} \bigg|_{t=0} \]
\[ = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
\[ = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \]

where \( c_1(t) \) and \( c_2(t) \) are the two coordinate functions of \( c(t) \).

5. section 2.5, #27
We write the function \( h(x) = \int_0^x f(x,y)dy \) as the composition of two functions \( \varphi_1: \mathbb{R} \rightarrow \mathbb{R}^2, \varphi_2: \mathbb{R}^2 \rightarrow \mathbb{R} \). Here \( g_1(x) = (x,x) \) and \( g_2(x,y) = \int_0^y f(x,t)dt \) where \( t \) is a dummy variable. Then we apply the chain rule to get
\[ h'(x) = Dg_2(g_1(x))Dg_1(x) \]

Now
\[ Dg_2 = \left( \frac{\partial g_2}{\partial x}, \frac{\partial g_2}{\partial y} \right) = \left( \int_0^y \frac{\partial f}{\partial x}(x,t)dt, f(x,y) \right) \]
So that \( Dg_2(g_1(x)) = Dg_2(x,x) = \int_0^x \frac{\partial f}{\partial x}(x,t)dt, f(x,x) \). Meanwhile
\[ Dg_1(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
so that
\[ h'(x) = \left( \int_0^x \frac{\partial f}{\partial x}(x,t)dt, f(x,x) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \int_0^x \frac{\partial f}{\partial x}(x,t)dt + f(x,x) \]
which is exactly what we wanted.
6. section 2.6, #2d
The directional derivative is
\[(y^2 + 3x^2y, 2xy + x^3)(4, -2) \cdot \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) = (-92, 48) \cdot \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) = \frac{52}{\sqrt{10}}\]

7. section 2.6, #4b
The normal to the surface is \((-2x, -2y, 0)\) evaluated at \((1, 2, 8)\) which is \((-2, -4, 0)\) so the tangent plane is
\[(x - 1, y - 2, z - 8) \cdot (-2, -4, 0) = 0\]
or\[x + 2y - 5 = 0\]

8. section 2.6, #8
A normal to the surface is \((3x^2y^3, 3x^3y^2 + 1, -1)\) evaluated at \((0, 0, 2)\) we gives \((0, 1, -1)\). This vector has length \(\sqrt{2}\) so normalizing it we get the unit normal vector \(1/\sqrt{2}(0, 1, -1)\).

9. section 2.6, #10
The level surfaces of the function \(f(x, y, z) = x^2 + y^2 + z^2\) are spheres centred at the origin. It is intuitively clear that going away from the origin in a straight line increases \(f\) fastest (in part because the picture is so symmetric). This direction at a point \((x_0, y_0, z_0)\) is precisely the vector \((x_0, y_0, z_0)\) which is (half of) \(\nabla f = (2x_0, 2y_0, 2z_0)\). This verifies theorem 13.

Similarly, the tangent plane to the sphere \(x^2 + y^2 + z^2 = constant\) at a point \((x_0, y_0, z_0)\) is perpendicular to the radius (the line from the origin to that point). Since this vector is \((x_0, y_0, z_0)\) it equals (half of) \(\nabla f\) again. This verifies theorem 14.