Instructions: You have 3 hours to complete this exam. You should work alone, without access to the textbook or class notes. You may not use a calculator. Do not discuss this exam with anyone except your instructor.

This exam consists of 9 questions. Except for the first problem (multiple choice), you must show your work to receive full credit. Be sure to indicate your final answer clearly for each question.

Note: If you use a major theorem (such as Green’s, Stokes’, or Gauss’ Divergence theorem), you must indicate it. Points will be deducted for failure to indicate the use of a major theorem. You must clearly indicate each time you use such a theorem.

Before turning in the exam, be sure to:

- Staple your exam with this cover sheet on top,
- Pledge your exam,
- Write your name and section number above.

The exam is due by Wednesday, May 10, 4 p.m. Good luck!

Pledge:

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1. In the following problem, let $S$ be a unit disk in the plane $z = 5$, centered at $(0, 0, 5)$, and oriented upward. Let $C$ be a straight path from $(2, 2, 2)$ to $(0, 0, 0)$. Also, let

\[
\begin{align*}
  f(x, y, z) &= xy^2z^3, \\
  F(x, y, z) &= -yi + xj, \\
  G(x, y, z) &= 3i + j + 2k.
\end{align*}
\]

Now, for each of the given quantities, decide if it is **positive**, **negative**, or **zero**. (You do not need to justify your answers. No partial credit will be given.)

(a) $\int\int_S \nabla \times F \cdot dS$.

(b) $\int_C \nabla f \cdot ds$.

(c) $\int\int_S F \cdot dS$.

(d) $\int\int_S G \cdot dS$.

(e) $\int_{\partial S} \nabla f \cdot ds$.

2. (a) Find all critical points of the function $g(x, y) = 2y^3 - 6y + x^2$ in $\mathbb{R}^2$ and classify them.

(b) Let $B$ be the unit ball in $\mathbb{R}^3$, i.e. the set of points $(x, y, z)$ satisfying $x^2 + y^2 + z^2 \leq 1$. Let $f(x, y, z) = 2x + 4y + 6z$. Find the minimum and maximum values of $f$ restricted to $B$.

3. (a) Let $f(x, y)$ be a $C^1$ function whose domain is the unit disk in the $xy$–plane, such that $f(x, y) \geq 0$ everywhere. Suppose that the level set $f(x, y) = 0$ is exactly the unit circle. Let $S$ be the graph of $f(x, y)$, oriented upward, and let

\[
F(x, y, z) = \text{curl}(x^2 - z, e^z + 2x, \pi).
\]

Determine $\int\int_S F \cdot dS$.

(b) Let $g(x, y)$ be a $C^1$ function whose domain is the unit disk in the $xy$–plane, such that $g(x, y) \leq 0$ everywhere. Suppose that the level set $g(x, y) = 0$ is exactly the unit circle. Let $T$ be the graph of $f(x, y)$, oriented downward.

Determine $\int\int_T F \cdot dS$.

4. Evaluate $\int_0^{\sqrt{3}} \int_{\sqrt{x}}^{\sqrt[3]{3}} x(\sqrt{1 + y^2}) \, dy \, dx$. Hint: Consider the region of integration.
5. Let $W$ be the region in space under the graph of

$$f(x, y) = (\cos y) \exp(1 - \cos 2x) + xy$$

over the region in the $xy$-plane bounded by the line $y = 2x$, the $x$ axis, and the line $x = \pi/4$.

(a) Find the volume of $W$.

(b) Let $\mathbf{F} = 5xi + 5yj + 5zk$ be the velocity field of a fluid in space. Calculate the flux of $\mathbf{F}$ through the boundary $\partial W$ of $W$, where $W$ is the region from (a).

6. Let $S$ be a surface in $\mathbb{R}^3$ given as follows. $S$ is the portion of the cylinder $x^2 + y^2 = 9$ lying above $z = 0$, below the graph of $z = \sqrt{x^2 + (y - 3)^2}$, and with $y \leq 0$.

(a) Set up, but do not evaluate, an integral giving the surface area of $S$.

(b) Suppose $S$ is made of a thin metal whose mass density at any point is given by the function $f(x, y, z) = \sqrt{1 - \frac{y}{3}}$. Find the total mass of $S$.

7. Let $g(x, y) = xe^y$.

(a) Compute the second-order Taylor formula for $g$ around $(3, 0)$.

(b) Approximate $2.9e^{0.1}$.

(c) Compute the directional derivative of $g$ based at $(3, 0)$ in the direction of fastest increase.

8. Let $\mathbf{F}(x, y) = 2xyi + x^2j$. Show that the line integral of $\mathbf{F}$ around the triangle $T$ (oriented counterclockwise) with vertices $(0, 0), (0, 1)$ and $(1, 1)$ is zero in the following three ways:

(a) parameterizing $T$ and evaluating the integral directly,

(b) showing $\mathbf{F}$ is a gradient vector field and explaining why the integral is zero, and

(c) using Green’s Theorem.

9. Compute the flux of the vector field

$$\mathbf{G}(x, y, z) = (xy^2, yz^2 + y, zx^2 + 1)$$

through the unit sphere, oriented outward.