1. Let \( f(x, y, z) \) be a scalar function, and let \( \mathbf{F}(x, y, z) \) be a vector field. (Assume both \( f \) and \( \mathbf{F} \) have continuous partial derivatives of all orders.) Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) be vectors in \( \mathbb{R}^3 \).

(a) \( \text{curl} \ \text{grad} \ f = \mathbf{0} \).

Solution: True.

(b) \( \text{div} \ \text{grad} \ f = 0 \).

Solution: False. For example if \( f(x, y, z) = x^2 \), then \( \text{grad} \ f = (2x, 0, 0) \) and \( \text{div} \ \text{grad} \ f = 2 \).

(c) \( \text{div} \ \text{curl} \ \mathbf{F} = 0 \).

Solution: True.

(d) Let \( C \) be an oriented curve. The path integral of \( f \) along \( C \) does not change when the orientation of \( C \) is reversed.

Solution: True.

(e) Let \( C \) be an oriented curve. The line integral of \( \mathbf{F} \) along \( C \) does not change when the orientation of \( C \) is reversed.

Solution: False. The integral changes by a minus-sign.

(f) The expression \( \mathbf{u} \cdot \mathbf{v} \) is a vector.

Solution: False.

(g) The expression \( \mathbf{u} \times \mathbf{v} \) is a vector.

Solution: True.

(h) The expression \( (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \) is a vector.

Solution: True. First note that \( (\mathbf{v} \cdot \mathbf{w}) \) is a scalar, so \( (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \) is the scalar product of the vector \( \mathbf{u} \) with the scalar \( (\mathbf{v} \cdot \mathbf{w}) \).

(i) Let \( S \) be an oriented surface. The quantity \( \iint_S \mathbf{F} \cdot d\mathbf{S} \) is a vector.

Solution: False.

(j) Let \( S \) be an oriented surface. The quantity \( \iint_S f \, dS \) is a vector.

Solution: False.
2. Let \( \mathbf{F}(x, y, z) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right) \).

(a) Show \( \text{curl } \mathbf{F} = (0, 0, 0) \).

(b) Let \( C \) be the unit circle in the \( xy \)-plane, oriented \textbf{clockwise}. Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{s} \).

(c) Using your answer from (b), explain why \( \mathbf{F} \) is not a gradient field, even though \( \text{curl } \mathbf{F} = (0, 0, 0) \).

Solution:

(a) This is a straight–forward computation:

\[
\text{curl } \mathbf{F} = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} = -i \frac{\partial}{\partial x} \frac{x}{x^2+y^2} + j \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} + k \left( \frac{\partial}{\partial x} \frac{x}{x^2+y^2} - \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} \right) = k \left( \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right) = (0, 0, 0).
\]

(b) We first have to parametrize \( C \). We take \( \mathbf{c}(t) = (\cos(-t), \sin(-t)), t \in [0, 2\pi] \).

(note that we need the \(-t\) to ensure that we go clockwise). We then compute

\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{t=0}^{t=2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_{t=0}^{t=2\pi} (-\sin(-t), \cos(-t), 0) \cdot (\sin(-t), -\cos(-t), 0) dt = \int_{t=0}^{t=2\pi} -1 dt = -2\pi.
\]

(Note that one could also choose the usual clockwise parametrization \( \mathbf{c}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi] \), and then changed the integral by a minus–sign.)

(c) If \( \mathbf{F} \) was a gradient field, then any line integral over a closed curve would be zero.

But in (b) we saw that this is not the case.

3. The surface \( S \) is parameterized by \( \Phi(u, v) = (e^u - 2, 2v + 3, 5 + u^2 + v^2) \) with \( u, v \in \mathbb{R} \).

(a) Determine the equation of the tangent plane to \((-1, 5, 6) \in S \).

(b) Find all points on \( S \) for which the tangent plane is parallel to the \( xy \)-plane.

Solution:
(a) We compute

\[ \mathbf{T}_u = (e^u, 0, 2u) \]
\[ \mathbf{T}_v = (0, 2, 2v) \]
\[ \mathbf{T}_u \times \mathbf{T}_v = (-4u, -2ve^u, 2e^u). \]

Now now note that \( \Phi(u, v) = (-1, 5, 6) \) for \( u = 0, \) \( v = 1 \) (which can be seen by looking at the first two coordinates).

For \( u = 0, \) \( v = 1 \) we have

\[ \mathbf{T}_u \times \mathbf{T}_v = (0, -2, 2). \]

This is the normal vector to the tangent plane at \((-1, 5, 6)\). The equation of the tangent plane is therefore given by

\[ (x - (-1), y - 5, z - 6) \cdot (0, -2, 2) = 0 \]

which simplifies to

\[ -2y + 2z - 2 = 0. \]

(b) First note that the tangent plane is parallel to the \( xy \)-plane if the \( z \)-components of \( \mathbf{T}_u \) and \( \mathbf{T}_v \) are zero. But this means that \( 2u = 0 \) and \( 2v = 0 \). So the only possibility is \( u = 0, v = 0 \). The corresponding point on \( S \) is \( \Phi(0, 0) = (-1, 3, 5) \).

4. Let \( f(x, y) = \frac{1}{3}x^3 + y\sqrt{2} + 3 \), and let \( D \) be the triangle with vertices \((0, 0)\), \((1, 0)\), and \((1, 1)\). Let \( S \) be the surface given by the graph of \( f(x, y) \) over \( D \).

(a) Find a parametrization of \( S \).

(b) Compute \( \int \int_S 4x^2 \, dS \).

Solution:

(a) The parametrization is given by

\[ \Phi(x, y) = (x, y, \frac{1}{3}x^3 + y\sqrt{2} + 3) \]

where the domain for \( x, y \) is given by \( D \), i.e. by the triangle with vertices \((0, 0)\), \((1, 0)\), and \((1, 1)\).

(b) Let’s first compute \( \|\mathbf{T}_x \times \mathbf{T}_y\|:\)

\[ \mathbf{T}_x = (1, 0, x^2) \]
\[ \mathbf{T}_y = (0, 1, \sqrt{2}) \]
\[ \mathbf{T}_x \times \mathbf{T}_y = (-x^2, -\sqrt{2}, 1) \]
\[ \|\mathbf{T}_x \times \mathbf{T}_y\| = \sqrt{3 + x^4}. \]

We compute

\[ \int \int_S 4x^2 \, dS = \int_{x=0}^{x=1} \int_{y=x}^{y=x} 4x^2 \sqrt{3 + x^4} \, dy \, dx = \int_{x=0}^{x=1} 4x^3 \sqrt{3 + x^4} \, dx. \]
Now let \( u = 3 + x^4 \) and we get
\[
\int_{x=0}^{x=1} 4x^3\sqrt{3 + x^4} \, dx = \int_{u=3}^{u=4} \sqrt{u} \, du = \left[ \frac{2}{3} u^{3/2} \right]_{u=3}^{u=4} = \frac{2}{3} (4^{3/2} - 3^{3/2}).
\]

5. Consider the solid hemisphere formed by taking the portion of the unit ball with \( y \geq 0 \). Let \( S \) be the surface of this region (so that \( S \) is a hemisphere, together with a flat ‘base’ in the \( xz \)-plane). Find the flux of the vector field \( \mathbf{V}(x, y, z) = -z \mathbf{i} + \mathbf{j} + x \mathbf{k} \) out of the surface \( S \).

You may find the following identity useful: \( \sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha) \).

Solution: We have to break this problem into two parts. We first integrate over the hemisphere, and then we integrate over the flat base.

For the hemisphere we take the parametrization
\[
\Phi(\theta, \phi) = (\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi))
\]
with \( \theta \in [0, \pi] \) and \( \phi \in [0, \pi] \). We compute
\[
\begin{align*}
\mathbf{T}_\theta &= (-\sin(\theta) \sin(\phi), \cos(\theta) \sin(\phi), 0) \\
\mathbf{T}_\phi &= (\cos(\theta) \cos(\phi), \sin(\theta) \cos(\phi), -\sin(\phi)) \\
\mathbf{T}_\theta \times \mathbf{T}_\phi &= (-\cos(\theta) \sin^2(\phi), -\sin(\theta) \sin^2(\phi), -\sin(\phi) \cos(\phi)) \\
&= -\sin(\phi) (\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)).
\end{align*}
\]

Note that \( \mathbf{T}_\theta \times \mathbf{T}_\phi \) points inward (this can be seen by considering the last line and noticing that \( -\sin(\phi) \) is negative). So it has the wrong orientation. We can fix this by putting a minus–sign into the formula. So the flux through the hemisphere is given by
\[
\begin{align*}
&= -\int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=\pi} \mathbf{V}(\Phi(\theta, \phi)) \cdot (\mathbf{T}_\theta \times \mathbf{T}_\phi) \, d\phi d\theta \\
&= -\int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=\pi} (-\cos(\phi), 1, \cos(\theta) \sin(\phi)) \cdot (-\cos(\theta) \sin^2(\phi), -\sin(\theta) \sin^2(\phi), -\sin(\phi) \cos(\phi)) \, d\phi d\theta \\
&= -\int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=\pi} -\sin(\theta) \sin^2(\phi) \, d\phi d\theta \\
&= -\int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=\pi} -\frac{1}{2} \sin(\theta)(1 - \cos(2\phi)) \, d\phi d\theta \\
&= -\int_{\theta=0}^{\theta=\pi} \left[ -\frac{1}{2} \sin(\theta)(\phi - \frac{1}{2} \sin(2\phi)) \right]_{\phi=0}^{\phi=\pi} \, d\theta \\
&= -\int_{\theta=0}^{\theta=\pi} -\frac{\pi}{2} \sin(\theta) \, d\theta \\
&= -\left[ -\frac{\pi}{2} \cos(\theta) \right]_{\theta=0}^{\theta=\pi} \\
&= \pi.
\]

Another way to approach this part of the problem is to see that \( x \mathbf{i} + y \mathbf{j} + z \mathbf{k} \) is the unit normal vector to the sphere at the point \((x, y, z)\). Thus \( \mathbf{V} \cdot \mathbf{n} = y \). To obtain the flux,
one integrates this (as a scalar surface integral) in spherical coordinates. (It is a little bit faster this way.)

Now let’s integrate over the base. The base lies in the $xz$–plane and the normal vector is $-\mathbf{j}$ (since outside is to the left), since the $\mathbf{j}$–component of $\mathbf{V}$ is constant one, we see that the flux integral is just $-1$ times the area of the base, which equals $-\pi$.

To compute the flux over $S$ we have to add up the results from the hemisphere and the base, and we get that the flux equals $\pi - \pi = 0$.

A more formal approach is to find a parametrization again. We could take

$$\Phi(r, \theta) = (r \cos(\theta), 0, r \sin(\theta))$$

where $r \in [0, 1]$ and $\theta \in [0, 2\pi]$. Then

$$T_r = (\cos(\theta), 0, \sin(\theta))$$

$$T_\theta = (-r \sin(\theta), 0, r \cos(\theta))$$

$$T_r \times T_\theta = (0, -r, 0).$$

This is the correct orientation, since $(0, -1, 0)$ points outside. So we have:

$$\int \int \mathbf{V} \cdot d\mathbf{S} = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \mathbf{V}(\Phi(r, \theta)) \cdot (T_r \times T_\theta) \, d\theta dr$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (-r \sin(\theta), 1, r \cos(\theta)) \cdot (0, -r, 0) \, d\theta dr$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} -r \, d\theta dr = -\pi.$$

6. Let $\mathbf{c}(t) = (1, -t^2, \cos t)$, $0 \leq t \leq \pi$. Evaluate

$$\int_\mathbf{c} \sin z \, dx - y^2 \, dy + 3xz \, dz.$$

Solution: We compute

$$\int_\mathbf{c} \sin z \, dx - y^2 \, dy + 3xz \, dz = \int_{t=0}^{t=\pi} \sin(\cos(t)) \frac{d}{dt}(1) - (-t^2)^2 \frac{d}{dt}(-t^2) + 3 \cos(t) \frac{d}{dt}(\cos(t)) \, dt$$

$$= \int_{t=0}^{t=\pi} 2t^5 - 3 \cos(t) \sin(t) \, dt$$

$$= \left[ \frac{t^6}{6} + \frac{1}{2} 3 \cos(t)^2 \right]_{t=0}^{t=\pi}$$

$$= \frac{\pi^6}{6}.$$