1. Find and classify all the critical points of 

\[ f(x, y) = \frac{1}{2}x^2 - xy + \frac{1}{3}y^3. \]

**Solution:** Since \( f \) is differentiable everywhere, the only critical points are the points where the gradient vanishes.

\[ \nabla f(x, y) = (x - y, y^2 - x) = (0, 0) \]

precisely at the points \((0, 0)\) and \((1, 1)\). Note \( \frac{\partial^2 f}{\partial x^2} = 1, \frac{\partial^2 f}{\partial y \partial x} = -1, \) and \( \frac{\partial^2 f}{\partial y^2} = 2y. \) Apply the second derivative test:

<table>
<thead>
<tr>
<th>point</th>
<th>( \frac{\partial^2 f}{\partial x^2} )</th>
<th>( D )</th>
<th>classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>1</td>
<td>-1</td>
<td>saddle</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>1</td>
<td>1</td>
<td>strict local min</td>
</tr>
</tbody>
</table>

2. Let \( g(x, y) = 2e^{-x} \cos y. \)

(a) Find the quadratic Taylor polynomial for \( g(x, y) \) around the point \((0, 0)\).

\[
\begin{align*}
g(0, 0) &= 2 \\
\frac{\partial g}{\partial x} \bigg|_{(0, 0)} &= -2e^{-x} \cos y \bigg|_{(0, 0)} = -2 \\
\frac{\partial^2 g}{\partial x^2} \bigg|_{(0, 0)} &= 2e^{-x} \cos y \bigg|_{(0, 0)} = 2 \\
\frac{\partial^2 g}{\partial y \partial x} \bigg|_{(0, 0)} &= -2e^{-x} \sin y \bigg|_{(0, 0)} = 0 \\
\frac{\partial g}{\partial y} \bigg|_{(0, 0)} &= -2e^{-x} \sin y \bigg|_{(0, 0)} = 0 \\
\frac{\partial^2 g}{\partial y^2} \bigg|_{(0, 0)} &= -2e^{-x} \cos y \bigg|_{(0, 0)} = -2
\end{align*}
\]

So the quadratic Taylor polynomial for \( g(x, y) \) around \((0, 0)\) is

\[ g(h_1, h_2) \approx 2 - 2h_1 + h_1^2 - h_2^2 \]

(b) Use your answer in part (a) to estimate \( 2e^{-0.2} \cos 0.4. \)

\[ We \ just \ evaluate \ g(0.2, 0.4) \approx 2 - 2(0.2) + (0.2)^2 - (0.4)^2 = 2 - .4 + .04 - .16 = 1.48. \]

\[ (The \ actual \ value \ is \ approximately \ 1.508202). \]

3. A tank is in the shape of a half-cylinder of radius 2 and height 3. It is situated in \( \mathbb{R}^3 \), given by the inequalities \( \sqrt{x^2 + y^2} \leq 2, y \geq 0, \) and \( 0 \leq z \leq 3. \) The temperature at the point \((x, y, z)\) is given by

\[ T(x, y, z) = 2yz^2\sqrt{x^2 + y^2} \degree C. \]

Find the average temperature in the tank.
Solution: We describe the tank in cylindrical coordinates as \(0 \leq r \leq 2\), \(0 \leq \theta \leq \pi\), and \(0 \leq z \leq 3\). Recall the formula \([T]_{av} = \frac{\iiint_W T(x,y,z) \, dV}{\iiint_W \, dV}\).

We use cylindrical coordinates to compute

\[
\iiint_W T(x,y,z) \, dV = \int_0^3 \int_0^\pi \int_0^2 (r \sin \theta)(z^2)(r) \, r \, dr \, d\theta \, dz
\]

\[
= 2 \left( \int_0^3 z^2 \, dz \right) \left( \int_0^\pi \sin \theta \, d\theta \right) \left( \int_0^2 r^3 \, dr \right)
\]

\[
= 2(2)(4)(9) = 144
\]

The denominator (volume of region) is given by the formula \(\text{Vol}(W) = \frac{\pi 2^2 3^2}{2} = 6\pi\), or by computing

\[
\iiint_W \, dV = \int_0^3 \int_0^\pi \int_0^2 r \, dr \, d\theta \, dz = \left( \int_0^3 dz \right) \left( \int_0^\pi \, d\theta \right) \left( \int_0^2 r \, dr \right) = (3)(\pi)(2) = 6\pi
\]

Thus, \([T]_{av} = \frac{144}{6\pi} = \frac{24}{\pi} \circ C\).

4. Let \(T\) be the triangle with vertices \((0,0), (1,1)\) and \((0,1)\) and let \(f(x,y) = x\sin(y^3)\).

(a) Find the correct limits of integration to set up \(\iint_T f(x,y) \, dA\) as a double integral

\[
\iint_T f(x,y) \, dx \, dy.
\]

Solution: \(\int_0^1 \int_y^0 f(x,y) \, dx \, dy\).

(b) Find the correct limits of integration to set up \(\iint_T f(x,y) \, dA\) as a double integral

\[
\iint_T f(x,y) \, dy \, dx.
\]

Solution: \(\int_0^1 \int_x^1 f(x,y) \, dy \, dx\).

(c) Compute \(\iint_T f(x,y) \, dA\).

Solution: Use the set up from (a):

\[
\iint_T f(x,y) \, dA = \int_0^1 \int_0^y x\sin(y^3) \, dx \, dy
\]

\[
= \int_0^1 \left[ \frac{\sin(y^3)}{2} \right]_0^y x \, dy
\]

\[
= \int_0^1 \frac{1}{2} y^2 \sin(y^3) \, dy
\]

\[
= \left[ \frac{\cos(y^3)}{6} \right]_0^1
\]

\[
= 1 - \cos \frac{1}{6}
\]
5. Find the maximum and minimum values obtained by \( f(x, y) = x + y^2 \) on the ellipse \( x^2 + 3y^2 \leq 9 \).

Solution: First, find critical points in the interior \( x^2 + 3y^2 < 9 \). Note \( \nabla f(x, y) = (1, 2y) \) is never \((0, 0)\), so there are no critical points in the interior.

Second, find critical points on the boundary \( x^2 + 3y^2 = 9 \) using Lagrange Multipliers. Our constraint function is \( g(x, y) = x^2 + 3y^2 \). Solve \( \nabla f(x, y) = \lambda \nabla g(x, y) \), i.e. \( (1, 2y) = \lambda (2x, 6y) \). The second coordinate gives two possibilities: \( y = 0 \) or \( \lambda = 1/3 \). If \( y = 0 \), then \( x = \pm 3 \) (from the constraint \( x^2 + 3y^2 = 9 \)). If \( \lambda = 1/3 \), then \( x = 3/2 \) (from \( 1 = 2\lambda x \)), and the constraint gives \( y = \pm 3/2 \). There are four critical points to investigate: \((\pm 3, 0)\) and \((3/2, \pm 3/2)\).

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((3, 0))</td>
<td>3</td>
</tr>
<tr>
<td>((-3, 0))</td>
<td>-3</td>
</tr>
<tr>
<td>((3/2, 3/2))</td>
<td>15/4</td>
</tr>
<tr>
<td>((3/2, -3/2))</td>
<td>15/4</td>
</tr>
</tbody>
</table>

Thus the absolute maximum value of \( f \) on boundary is \( 15/4 \), and the absolute minimum value on the boundary is \(-3\).

Since there are no critical points from the interior, these maximum and minimum boundary values are also the maximum and minimum values throughout the entire region.

6. The region \( S \) is cut from a solid ball of radius 1 centered at the origin. \( S \) is the region cut by the inequalities \( z \geq 0 \) and \( y \geq x \). \( S \) is one-quarter of the entire ball, and contains the point \((0, 1, 0)\).

The mass density of \( S \) at a point \((x, y, z)\) is given by the function \( \delta(x, y, z) = 30z^2 \text{ kg/m}^3 \).

(a) Find the total mass of \( S \).

Solution: The total mass is given by \( \iiint_S \delta(x, y, z) \, dV \). Note that \( S \) is described in spherical coordinates by \( 0 \leq \rho \leq 1, \pi/4 \leq \theta \leq 5\pi/4, \) and \( 0 \leq \phi \leq \pi/2 \). Thus

\[
\iiint_S \delta(x, y, z) \, dV = \int_0^{\pi/2} \int_{\pi/4}^{5\pi/4} \int_0^1 30(\rho \cos \phi)^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

\[
= 30 \left( \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \right) \left( \int_{\pi/4}^{5\pi/4} d\theta \right) \left( \int_0^1 \rho^4 \, d\rho \right)
\]

\[
= 30 \left( -\frac{\cos^3 \phi}{3} \right) \bigg|_{\pi/2}^{\pi/4} \left( \frac{1}{5} \right)
\]

\[
= 2\pi \text{ kg}
\]

(b) Find the average mass density of \( S \).

Solution: Average mass density is

\[
[\delta]_{av} = \frac{\iiint_S \delta(x, y, z) \, dV}{\iiint_S \, dV} = \frac{2\pi \text{ kg}}{\pi/3 \text{ m}^3} = \frac{2\pi \text{ kg}}{\pi/3 \text{ m}^3} = 6 \text{ kg/m}^3
\]