Math 212 Multivariable Calculus - Midterm II
Due November 12th, 2001

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have two hours. Show all your work for a full credit. If more space is needed, use the back pages.

Print name: ________________________________

Upon finishing please sign the pledge below:

On my honor I have neither given nor received any aid on this exam, and have observed the time limit given. I started working on this exam at __ __ and finished at __ __ on the __th day of November.

Signature: ________________________________

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(1) Let \( c(t) \) be a path such that \( c'(t) \) is never 0. The \emph{curvature} of \( c \) at time \( t \) is given by
\[
\frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}.
\]
It measures how \emph{sharp} the turn of the path is at the given time.

(a) Let \( c(t) = (2 \cos t, \sin t, 0) \). Find the points of the highest curvature and the points of the lowest curvature.

\textbf{Solution} \( c'(t) = (-2 \sin t, \cos t, 0) \) and \( c''(t) = (-2 \cos t, -\sin t, 0) \). Hence
\[
c'(t) \times c''(t) = (2 \sin^2 t + 2 \cos^2 t) \mathbf{k} = 2 \mathbf{k}.
\]
Therefore curvature at \( t \) is
\[
\frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3} = \frac{2}{(4 \sin^3 t + \cos^2 t + \cos^2 t)^{3/2}} = \frac{2}{(3 \sin^2 t + 1)^{3/2}}.
\]
\textit{Minimum} occurs when \( \sin^2 t = 1 \). i.e. when \( t = \pi/2, 3\pi/2 \). i.e. at \((0, \pm 1, 0)\).
\textit{Maximum} occurs when \( \sin^2 t = 0 \). i.e. when \( t = 0, \pi \). i.e. at \((\pm 2, 0, 0)\).
Draw the image of \( c(t) \) and see that it turns sharp at \((\pm 2, 0, 0)\) and slow at \((0, \pm 1, 0)\).

(b) Answer the question in (a) for \( c(t) = (2 \cos t, \sin t, \sin t) \).

\textbf{Solution} \( c'(t) = (-2 \sin t, \cos t, \cos t) \) and \( c''(t) = (-2 \cos t, -\sin t, -\sin t) \). Hence
\[
c'(t) \times c''(t) = -2 \mathbf{j} + 2 \mathbf{k}.
\]
Therefore curvature at \( t \) is
\[
\frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3} = \frac{2 \sqrt{2}}{(4 \sin^3 t + \cos^2 t + \cos^2 t)^{3/2}} = \frac{2 \sqrt{2}}{(2 \sin^2 t + 2)^{3/2}}.
\]
\textit{Minimum} occurs when \( \sin^2 t = 1 \). i.e. when \( t = \pi/2, 3\pi/2 \). i.e. at \((0, \pm 1, \pm 1)\).
\textit{Maximum} occurs when \( \sin^2 t = 0 \). i.e. when \( t = 0, \pi \). i.e. at \((\pm 2, 0, 0)\).
(2) Consider the vector field $\mathbf{F}(x, y) = -yi + xj$.

(a) Sketch the vector field, particularly at the following eight points: $(1, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(0, 1)$, $(-1/\sqrt{2}, 1/\sqrt{2})$, $(-1, 0)$, $(-1/\sqrt{2}, -1/\sqrt{2})$, $(0, -1)$, $(1/\sqrt{2}, -1/\sqrt{2})$.

(b) Draw the flow line through the 8 points given above.

Solution It is the unit circle centered at the origin.

(c) Find an equation of the flow line $c(t)$ for $\mathbf{F}$ through $(1, 0)$. Verify that $\mathbf{F}(c(t)) = c'(t)$.

Solution $c(t) = (\cos t, \sin t)$. $\mathbf{F}(c(t)) = (-\sin t, \cos t) = c'(t)$. 


(3) Let $\mathbf{F}(x, y, z) = -\nabla V$ and $\mathbf{c}(t)$ be a flow line for $\mathbf{F}$. Prove that $V(\mathbf{c}(t))$ is everywhere decreasing. (i.e. its derivative with respect to $t$ is always negative.) Explain why geometrically. (*Hint*: recall that the direction of the gradient of a function has a special meaning.)

**Solution** By the chain rule, we have

$$\frac{d}{dt} V(\mathbf{c}(t)) = \nabla V(\mathbf{c}(t)) \cdot \mathbf{c}'(t).$$

But since $\mathbf{c}$ is a flow line for $\mathbf{F}$, $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)) = -\nabla (\mathbf{c}(t))$ and

$$\nabla V(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \nabla V(\mathbf{c}(t)) \cdot (-\nabla (\mathbf{c}(t))) = -\| \nabla V(\mathbf{c}(t)) \|^2 \leq 0.$$

Geometrically: gradient points to the direction in which the function increases the fastest. Since $\mathbf{c}'$ is the negative of the gradient, $\mathbf{c}$ actually traces out the path of steepest descent.
(4) Compute the curl of the following vector field.

\[ \mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) \]

**Solution** Let \( f(x, y, z) = x^2 + y^2 + z^2 \) and \( \mathbf{F}_0 = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \). We use the formula

\[ \nabla \times (f \mathbf{F}_0) = \nabla f \times \mathbf{F}_0 + f \nabla \times \mathbf{F}_0. \]

Since \( \mathbf{F}_0 \) is a constant vector field, \( \nabla \times \mathbf{F}_0 = 0 \) and the above equals \( \nabla f \times \mathbf{F}_0 \). Now, \( \nabla f = (2x, 2y, 2z) \) and

\[ \nabla f \times \mathbf{F}_0 = 2x \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x & 2y & 2z \\ 3 & 4 & 5 \end{vmatrix} = (10 - 8z, 6z - 10x, 8x - 6y). \]
(5) Consider the function \( f(x, y) = x^2 + xy + y^2 \) on the unit disk \( D = \{(x, y) \mid x^2 + y^2 \leq 1\} \).

(a) Find the absolute maximum and the absolute minimum of \( f(x, y) \) on \( D \).

**Solution**

1. **Interior**: \( Df = [2x + y x + 2y] = 0 \) only when \((x, y) = (0, 0)\). Hence \((0, 0)\) is the only critical point and \( f(0, 0) = 0 \).
2. **Boundary**: On the boundary \( x^2 + y^2 = 1 \), we have \( f(x, y) = x^2 + xy + y^2 = xy + 1 \) and this function takes maximum when \( x = y = \pm \frac{1}{\sqrt{2}} \), \( f(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}) = \frac{3}{2} \) and minimum when \( x = -y \), \( f(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}) = \frac{1}{2} \).
3. **Compare numbers**: Absolute maximum is \( \frac{3}{2} \) and the absolute minimum is 0.

(b) State the Mean Value Theorem for double integrals.

**Solution** See the textbook.

(c) Using the Mean Value Theorem, prove the following inequality:
\[
\pi \leq \iint_D \exp (f(x, y)) \, dA \leq \pi e^{3/2}.
\]

**Solution** By (a), \( 0 \leq f(x, y) \leq \frac{3}{2} \). Hence \( 1 \leq e^{f(x,y)} \leq e^{3/2} \) By the MVT,
\[
1 \cdot \text{Area}(D) \leq \iint_D \exp (f(x, y)) \, dA \leq e^{3/2} \text{Area}(D).
\]
But \( \text{Area}(D) = \pi \) and we are done.
(6) Compute the volume of the solid bounded by the surface \( z = \sin y \), the planes \( x = 1 \), \( x = 0 \), \( y = 0 \), \( y = \pi/2 \) and the \( xy \)-plane.

**Solution**  Simply, this is just \( \int_0^1 \int_0^{\pi/2} \sin y \, dy \, dx \). The answer is 1.
(7) Let $W$ be the pyramid with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$. Express $W$ as an elementary region by completing the following steps.

(a) Fix nothing: what are the maximum and minimum of $z$?

   Solution  $0 \leq z \leq 1$.

(b) Fix $z$: what are the maximum and minimum of $y$?

   Solution  Equation of the plan containing $(0,0,1)$, $(1,1,0)$, $(0,1,0)$ is $y = 1 - z$.

   Therefore,

   \[ 0 \leq y \leq 1 - z. \]

(c) Fix $z$ and $y$: what are the maximum and minimum of $x$?

   Solution  Equation of the plan containing $(0,0,1)$, $(1,1,0)$, $(1,0,0)$ is $x = 1 - z$.

   Therefore,

   \[ 0 \leq x \leq 1 - z. \]
(8) (Note: This problem has been slightly modified.) Let \( W \) be the region bounded by the cone \( z = \sqrt{x^2 + y^2} \) and the cylinder \( x^2 + y^2 = 1 \), and the \( xy \)-plane. Evaluate the following integral.

\[
\iiint_W \exp(x^2 + y^2)^{3/2} dV.
\]

**Solution** Use the cylindrical coordinate.

\[
\int_0^{2\pi} \int_0^1 \int_0^r e^{r^3} r dz dr d\theta = 2\pi \int_0^1 r^2 e^{r^3} dr = 2\pi \int_0^1 e^{ru} du = \frac{2\pi}{3} (e - 1).
\]
(9) Let $W$ be the region bounded by $x + y + z = 1$, $x = 0$, $y = 0$ and $z = 0$. Evaluate the following integral.
\[ \int \int \int_W (x^2 + y^2 + z^2) \, dx \, dy \, dz. \]
(10) Let $P$ be the parallelogram bounded by $y = 2x$, $y = 2x - 2$, $y = x$, and $y = x + 1$. Evaluate
\[ \int \int_P xy \, dx \, dy \]
by completing the following steps.

(a) Find a one-to-one and onto map $T$ that maps $P^* = [0, 1] \times [0, 1]$ to $P$.

(b) Evaluate the integral using $T$ and the theorem of change of variables.
(11) Evaluate \( \int \int \int_{W} \exp \left( (x^2 + y^2 + z^2)^{3/2} \right) \, dV \) for the following \( W \)'s.

(a) \( W \) is the upper half of the unit ball. i.e. \( W = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \geq 0 \} \).

(b) \( W \) is the region bounded by the unit sphere \( x^2 + y^2 + z^2 = 1 \) and the upper cone \( z = \sqrt{x^2 + y^2} \).