1. Let \( P = (2, 3, 1), Q = (2, 2, 2), R = (3, 3, -1) \).

(a) Find the equation of the plane through \( P, Q \) and \( R \).

Solution: We see that the plane is parallel to the vectors \( v = Q - P = (0, -1, 1) \) and \( w = R - P = (1, 0, -2) \) hence the normal direction to the plane is \( v \times w = (2, 1, 1) \). Therefore, we conclude that the plane is the set of points \( X = (x, y, z) \) which satisfy
\[
0 = (X - P) \cdot (v \times w) = 2(x - 2) + (y - 3) + (z - 1) = 2x + y + z - 8.
\]

(b) Let \( l \) be the line given by \( x = 2 - t, y = 3t, z = 1 + 2t \). Find the intersection point of the plane and the line.

Solution: a point of intersection must lie on both the plane and the line, and thus satisfies the equations given and the equation from (a). We substitute to obtain:
\[
0 = 2(2 - t) + 3t + 1 + 2t - 8 = 3t - 3.
\]
Thus we must have \( t = 1 \). This corresponds to the point \( (1, 3, 3) \).

2. Let \( P \) be the plane given by the equation \( x_1 - 2x_2 + 3x_3 + 4 = 0 \) and let \( Q \) be the plane given by all points of the form
\[
(2, 4, 1) + \lambda (3, 3, 1) + \mu (2, -1, 1), \quad \lambda, \mu \in \mathbb{R}.
\]
Determine whether \( P \) and \( Q \) are parallel or not.

Note: Two planes are called parallel if they do not intersect, an equivalent condition is that two planes are parallel if their normal vectors are parallel.

Solution: A vector normal to \( P \) can be found by thinking of \( P \) as the 0-level set for the function \( f(x_1, x_2, x_3) = x_1 - 2x_2 + 3x_3 + 4 \) and taking the gradient at some convenient point. This is \( n_1 = \nabla f = (1, -2, 3) \). This is at any point, as the gradient is constant.

The normal vector to \( Q \) is \( n_2 = (3, 3, 1) \times (2, -1, 1) = (4, -1, -9) \). It is pretty clear that \( n_1 \) and \( n_2 \) are not parallel (they are not proportional), so neither are \( P \) and \( Q \).

3. Consider the two functions \( f : \mathbb{R} \to \mathbb{R}^3, f(t) = (\cos(t), \sin(t), t), g : \mathbb{R}^3 \to \mathbb{R}, g(x, y, z) = x^2 + y^2 + z^2 \).

(a) Use the chain rule to compute the derivative of \( g \circ f \) at the point \( t = \pi/4 \).

Solution: We compute that \( f(\pi/4) = (\sqrt{2}/2, \sqrt{2}/2, \pi/4), \) and
\[
Df(\pi/4) = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 1 \end{pmatrix},
Dg(\sqrt{2}/2, \sqrt{2}/2, \pi/4) = (\sqrt{2} \sqrt{2} \pi/2),
\]
and finally that

\[ D(g \circ f)(\pi/4) = Dg(\sqrt{2}/2, \sqrt{2}/2, \pi/4)Df(\pi/4) = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \pi/2 \end{pmatrix} \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 1 \end{pmatrix} = \pi/2. \]

**Alternate Solution:** The chain rule says that

\[ \frac{d(g \circ f)}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} + \frac{\partial g}{\partial z} \frac{dz}{dt}. \]

Then we compute the required partial derivatives and substitute in, recalling that \( f(t) = (x(t), y(t), z(t)) \).

(b) View the function \( f \) as describing a path in 3-space. Write an equation for the tangent line to this path at the point \((0, 1, \pi/2)\).

**Solution:** The line is those points described by

\[ (0, 1, \pi/2) + t(-1, 0, 1), \quad t \in \mathbb{R}. \]

4. An astronaut is floating in the middle of a nebula in outer space. The gas in this nebula is very hot, and she must decrease the temperature she experiences as quickly as possible. In a rectangular coordinate system centered on her, the temperature of the gas (in degrees Centigrade) is described by the equation

\[ T(x, y, z) = -2x + \sin(x^2)y^2 + 2z + 78. \]

(a) Which direction should the astronaut go? (Note that the astronaut is located at \((0, 0, 0)\)).

**Solution:** the astronaut should go in the direction of greatest decrease of the function, which is the negative of the gradient. So we compute \( \nabla T(0, 0, 0) = (-2, 0, 2) \) and suggest that the astronaut go in the direction of \(-\nabla T(0, 0, 0) = (2, 0, -2)\)

(b) Would traveling in the direction of the vector \( \mathbf{v} = (-1, -17, 1) \) increase or decrease the temperature she experiences?

**Solution:** We need to compute the directional derivative in the direction specified. It is

\[ \nabla T(0, 0, 0) \cdot (-1, -17, 1)/||v|| = 4/||v|| > 0. \]

As this is positive, the temperature will increase in this direction.
5. Consider the graph of the function

\[ f(x, y) = x^3(y^2 - 1) + (x - y)^3. \]

(a) Find the equation of the tangent plane to the graph at \( P = (1, 2). \)

Solution: Can do this by just recalling the linear approximation formula. Another way follows. The graph of \( f \) is the 0-level set of the function

\[ g(x, y, z) = f(x, y) - z = x^3(y^2 - 1) + (x - y)^3 - z. \]

Thus the tangent plane can be recovered as the vectors orthogonal to the gradient of \( g \). So we compute

\[ \nabla g(1, 2, 2) = (12, 1, -1) \]

and find the plane as those points \( X = (x, y, z) \) which satisfy

\[ 0 = (X - (1, 2, 2)) \cdot (12, 1, -1) = 12x + y - z - 12. \]

(b) Find a unit vector which is normal to the graph at \( P \).

Solution: The above solution already finds a normal vector for us as \( \nabla g(1, 2, 2) = (12, 1, -1) \). We need merely normalize it to find a unit vector. The length of the gradient is \( \sqrt{146} \), so the vector we want is \( \left( \frac{12}{\sqrt{146}}, \frac{1}{\sqrt{146}}, -\frac{1}{\sqrt{146}} \right) \).

6. (a) Consider \( h(x, y) = x^y \). Find the partial derivatives \( \frac{\partial h}{\partial x} \) and \( \frac{\partial h}{\partial y} \).

Solution: We compute that

\[ \frac{\partial h}{\partial x} = yx^{y-1}, \quad \text{and} \quad \frac{\partial h}{\partial y} = (\ln x)x^y. \]

(b) Use (a) and the Chain Rule to find

\[ \frac{d}{dt} \left( f(t)^{g(t)} \right). \]

Solution: Let \( z(t) = (f(t), g(t)) \). Then \( f(t)^{g(t)} = h \circ z(t) \). So by (a) and the chain rule, we get that

\[ \frac{d}{dt}(h \circ z) = \frac{\partial h}{\partial x}(f(t), g(t)) \frac{df}{dt} + \frac{\partial h}{\partial y}(f(t), g(t)) \frac{dg}{dt} = g(t)[f(t)]^{g(t)-1} f'(t) + \ln(f(t))[f(t)]^{g(t)} g'(t). \]