Math 211 Fall 2007: Solutions: HW #7
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1. section 4.5, #18
The characteristic polynomial is $\lambda^2 + 3\lambda + 2 = 0$ so $(\lambda + 1)(\lambda + 2) = 0$ and the general solution to the homogeneous equation is $A e^{-t} + B e^{-2t}$.

A particular solution should have the form $y_p = Ce^{-4t}$. Then $y_p' = -4Ce^{-4t}$ and $y_p'' = 16Ce^{-4t}$. Plugging back into the ODE we get

$$16Ce^{-4t} + 3(-4Ce^{-4t}) + 2Ce^{-4t} = 3e^{-4t}$$

which gives

$$16C - 12C + 2C = 3$$

or $C = 1/2$. So anything of the form $y = Ae^{-t} + Be^{-2t} + \frac{1}{2}e^{-4t}$ is a solution.

The initial condition $y(0) = 1$ means $A + B + \frac{1}{2} = 1$. Since $y'(0) = -Ae^{-t} - 2Be^{-2t} + \frac{1}{2}(-4)e^{-4t}$ the condition $y'(0) = 0$ means $-A - 2B = 0$. Solving for $A$ and $B$ gives $B = \frac{3}{2}$ and $A = 3$. So we get the solution

$$y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}.$$ 

2. section 4.5, #21
The characteristic polynomial is $\lambda^2 - 2\lambda + 5 = 0$ which has roots $\lambda_1 = 1+2i$ and $\lambda_2 = 1 - 2i$. This leads to the homogeneous solution

$$e^t(A \cos(2t) + B \sin(2t)).$$

A particular solution will look like $y_p = C \cos(t) + D \sin(t)$. Then $y_p' = -C \sin(t) + D \cos(t)$ and $y_p'' = -C \cos(t) - D \sin(t)$. Plugging back gives

$$(-C \cos(t)-D \sin(t)) - 2(-C \sin(t)+D \cos(t)) + 5(C \cos(t) + D \sin(t)) = 3 \cos(t)$$

which simplifies to

$$\cos(t)(-C - 2D + 5C - 3) + \sin(t)(-D + 2C + 5D) = 0$$

which means $4C - 2D = 3$ and $4D + 2C = 0$. So $C = 6/10$ and $D = -3/10$. The general solution is

$$y = e^t(A \cos(2t) + B \sin(2t)) + 3/5 \cos(t) - 6/10 \sin(t)$$

Initial condition $y(0) = 0$ means $A + 3/5 = 0$. So $A = -3/5$.

$$y' = e^t(A \cos(2t)+B \sin(2t))+e^t(-2A \sin(2t)+B2 \cos(2t))−6/10 \sin(t)−3/10 \cos(t)$$

3. section 4.5, #22
The characteristic polynomial is $\lambda^2 + 4\lambda + 4 = 0$ so $\lambda = -2$ is a double root. Hence general homogeneous solution is

$$e^{-2t}(A + Bt)$$

Particular solution has the form $y_p = C + Dt$. Then $y'_p = D$ and $y''_p = 0$. Plugging in we get

$$0 + 4(D) + 4(C + Dt) = 4 - t$$

which means $4C + 4D = 4$ and $4D = -1$. So general solution is

$$y = e^{-2t}(A + Bt) - 1/4t + 5/4$$

$y(0) = -1$ implies $A + 5/4 = -1$ so $A = -9/4$. $y' = -2e^{-2t}(A + Bt) + e^{-2t}(B) - 1/4$ so $y'(0) = 0$ implies $-2A + B - 1/4 = 0$ and so $B = -17/4$.

4. section 4.5, #26
Both $\cos(2t)$ and $\sin(2t)$ are solutions of homogeneous equations so we look for solution of the form

$$y_p = At \cos(2t) + Bt \sin(2t).$$

Then

$$y'_p = A \cos(2t) + B \sin(2t) + t(-2A \sin(2t) + 2B \cos(2t))$$

$$y''_p = -2A \sin(2t) + 2B \cos(2t) + (-2A \sin(2t) + 2B \cos(2t)) + t(-4A \cos(2t) - 4B \sin(2t))$$

Plugging back we get

$$-2A \sin(2t) + 2B \cos(2t) + (-2A \sin(2t) + 2B \cos(2t)) + t(-4A \cos(2t) - 4B \sin(2t)) + 4(At \cos(2t) + Bt \sin(2t))$$

which simplifies to give

$$\cos(2t)(4B - 4) + \sin(2t)(-4A) = 0$$

meaning $B = 1$ and $A = 0$ so a solution is $y_p = t \sin(2t)$.

5. section 4.6, #2
The homogeneous ODE has solutions $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$ so we look for a solution $y = v_1 y_1 + v_2 y_2$. Now

$$y' = v'_1 \cos(2t) + v'_2 \sin(2t) - 2v_1 \cos(2t) + 2v_2 \sin(2t)$$

To simplify calculations we restrict that $v'_1 \cos(2t) + v'_2 \sin(2t) = 0$ so $y' = -2v_1 \cos(2t) + 2v_2 \sin(2t)$ and

$$y'' = -2v'_1 \cos(2t) + 2v'_2 \sin(2t) - 4(v_1 \cos(2t) + v_2 \sin(2t))$$
Then
\[ y'' + 4y = -2v'_1 \cos(2t) + 2v'_2 \sin(2t) = \sec(2t) \]
Solving for \( v'_1 \) and \( v'_2 \) we get \( v'_1 = -\frac{1}{2} \tan(2t) \) and \( v'_2 = 1/2 \). Integrating we get \( v_1 = \frac{1}{4} \ln(\cos(2t)) \) and \( v_2 = t/2 \). So solution is
\[ y = \frac{1}{4} \cos(2t) \ln(\cos(2t)) + \frac{t}{2} \sin(2t) \]

6. section 4.6, #6
The solutions of the homogeneous ODE are \( x_1 = e^{2t} \) and \( x_2 = te^{2t} \) so we look for a solution of the form
\[ x(t) = v_1 x_1 + v_2 x_2 \]

We proceed just as above (details omitted) and find that \( v'_1 = -t \) and \( v'_2 = 1 \) so \( v_1 = -t^2/2 \) and \( v_2 = t \) and \( x(t) = \frac{1}{2}t^2e^{2t} \).

7. section 4.6, #13
Plugging in we check that \( y_1 = t \) and \( y_2 = t^{-3} \) are indeed solutions. So we look for a general solution
\[ y = v_1 t + v_2 t^{-3} \]

First divide the ODE by \( t^2 \) to get the standard form \( y'' + 3/ty' - 3/t^2y = 1/t^3 \). Solving as above we get \( v'_1 = \frac{1}{4}t^{-3} \) so \( v_1 = -1/8t^{-2} \) and \( v'_2 = -1/4t \) so \( v_2 = -1/8t^2 \). So \( y_p = v_1 y_1 + v_2 y_2 = -\frac{1}{4t} \) is a solution and the general solution is
\[ At + Bt^{-3} - \frac{1}{4t} \]

NOTE: you can use formulas (6.16) in the book to determine \( v_1 \) and \( v_2 \) but only after you are sure you know, understand and are comfortable finding the solution without the formulas (as illustrated in 4.6 #2 above).