1. section 9.1, #16
The characteristic polynomial is
\[ \det(A - \lambda I) = (2 - \lambda)(-2 - \lambda) \]
so the eigenvalues are \( \lambda_1 = 2 \) and \( \lambda_2 = -2 \). The eigenvector for \( \lambda_1 \) satisfies
\[ \begin{pmatrix} 0 & 0 \\ -4 & -4 \end{pmatrix} v_1 = 0 \]
so that \( v_1 = (1, -1)^t \). The eigenvector for \( \lambda_2 \) satisfies
\[ \begin{pmatrix} 4 & 0 \\ -4 & 0 \end{pmatrix} v_2 = 0 \]
so \( v_2 = (0, 1)^t \).
So a fundamental set of solutions is formed by \( e^{2t} (1, -1)^t \) and \( e^{-2t} (0, 1)^t \).

2. section 9.1, #22
The characteristic polynomial is
\[ \det(A - \lambda I) = (-3 - \lambda)(4 - \lambda) \]
so the eigenvalues are \( \lambda_1 = -3 \) and \( \lambda_2 = 4 \). The eigenvector for \( \lambda_1 \) satisfies
\[ \begin{pmatrix} 0 & 14 \\ 0 & 7 \end{pmatrix} v_1 = 0 \]
so that \( v_1 = (1, 0)^t \). The eigenvector for \( \lambda_2 \) satisfies
\[ \begin{pmatrix} -7 & 14 \\ 0 & 0 \end{pmatrix} v_2 = 0 \]
so \( v_2 = (2, 1)^t \).
So a fundamental set of solutions is formed by \( e^{-3t} (1, 0)^t \) and \( e^{4t} (2, 1)^t \).

3. section 9.1, #24
The characteristic polynomial is
\[ \det(A - \lambda I) = (-5 - \lambda)(-3 - \lambda)(5 - \lambda) - 6(-4(-3 - \lambda)) = -\lambda^3 - 3\lambda^2 + \lambda + 3 = -(\lambda+3)(\lambda-1)(\lambda+1) \]
so the eigenvalues are \( \lambda_1 = -3 \), \( \lambda_2 = 1 \) and \( \lambda_2 = -1 \). The eigenvector for \( \lambda_1 \) satisfies
\[ \begin{pmatrix} -2 & 0 & -6 \\ 26 & 0 & 38 \\ 4 & 0 & 8 \end{pmatrix} v_1 = 0 \]
so that \( v_1 = (0,1,0)^t \). The eigenvector for \( \lambda_2 \) satisfies
\[
\begin{pmatrix}
-6 & 0 & -6 \\
26 & -4 & 38 \\
4 & 0 & 4 
\end{pmatrix}
\begin{pmatrix}
v_2 \\
1 \\
0 
\end{pmatrix} = 0
\]
so that \( v_2 = (1,0,-1)^t \). The eigenvector for \( \lambda_3 \) satisfies
\[
\begin{pmatrix}
-4 & 0 & -6 \\
26 & -2 & 38 \\
4 & 0 & 6 
\end{pmatrix}
\begin{pmatrix}
v_3 \\
1 \\
0 
\end{pmatrix} = 0
\]
so that \( v_3 = (3,0,-2)^t \).

So a fundamental set of solutions is formed by \( e^{3t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) and \( e^t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \) and \( e^{-t} \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \).

4. section 9.2, #6

The characteristic polynomial is
\[
\det(A - \lambda I) = (-1 - \lambda)(-1 - \lambda) - 1 = \lambda^2 + 2\lambda
\]
so the eigenvalues are \( \lambda_1 = 0 \) and \( \lambda_2 = -2 \). The eigenvector for \( \lambda_1 \) is \( v_1 = (1,1)^t \). The eigenvector for \( \lambda_2 \) satisfies
\[
\begin{pmatrix}
1 & 1 \\
1 & 1 
\end{pmatrix}
\begin{pmatrix}
v_2 \\
1 
\end{pmatrix} = 0
\]
so \( v_2 = (1,-1)^t \). So the general solution is
\[
C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

5. section 9.2, #14

\( e^{(1+i)t} = e^t (\cos(t) + i \sin(t)) \). So \( e^{(1+i)t}(-1 + i) = e^t (-\cos(t) + i \cos(t) - i \sin(t) - \sin(t)) \). Thus the real part of \( z(t) \) is
\[
\begin{pmatrix}
e^t(-\cos(t) - \sin(t)) \\
2e^t \cos(t) 
\end{pmatrix}
\]
and the imaginary part is
\[
\begin{pmatrix}
e^t(\cos(t) - \sin(t)) \\
2e^t \sin(t) 
\end{pmatrix}
\]
6. section 9.2, #16

The characteristic polynomial is

$$\det(A - \lambda I) = (-4 - \lambda)(4 - \lambda) + 32 = \lambda^2 + 16$$

so the two eigenvalues are $\lambda_1 = 4i$ and $\lambda_2 = -4i$. The eigenvector for $\lambda_1$ satisfies

$$\begin{pmatrix} -4 & -8 \\ 4 & 4i \end{pmatrix}v_1 = 0$$

so $v_1 = (-1, \frac{1+i}{2})^t$. Hence the other eigenvector is the conjugate of $v_1$, namely $v_2 = (-1, \frac{1-i}{2})^t$. A fundamental set of real solutions is given by the real and imaginary parts of

$$e^{4it} \begin{pmatrix} -1 \\ i \end{pmatrix}$$

Since $e^{4it} \frac{1+i}{2} = (\cos(4t) + i \sin(4t))\frac{1+i}{2} = \frac{1}{2}(\cos(4t) + i \sin(4t) + i \cos(4t) - \sin(4t))$ the real part is

$$\begin{pmatrix} -\cos(4t) \\ \frac{1}{2}(\cos(4t) - \sin(4t)) \end{pmatrix}$$

while the imaginary part is

$$\begin{pmatrix} -\sin(4t) \\ \frac{1}{2}(\sin(4t) + \cos(4t)) \end{pmatrix}$$

7. section 9.2, #32

The characteristic polynomial is

$$\det(A - \lambda I) = (-2 - \lambda)(2 - \lambda) + 4 = \lambda^2$$

so there’s only one eigenvalue $\lambda = 0$. An eigenvector is $v_1 = (1, -2)^t$. To find the other generalized eigenvector $v_2$ we must solve $Av_2 = v_1$ which gives $v_2 = (1, -3)^t$. Hence the general solution is a linear combination $C_1y_1 + C_2y_2$ of $y_1(t) = v_1 = (1, -2)^t$ and $y_2(t) = tv_1 + v_2 = (t+1, -2t-3)^t$.

8. section 9.2, #58

a) Denote by $y_1(t)$ and $y_2(t)$ the salt content in the top and bottom tanks at time $t$. Then

$$y'_1 = salt in - salt out = 0 - 5 \cdot \frac{y_1}{500}$$

while

$$y'_2 = salt in - salt out = 5 \cdot \frac{y_1}{500} - 5 \cdot \frac{y_2}{500}$$
Hence the initial value problem is $y' = Ay$ where

$$A = \begin{pmatrix} -1/100 & 0 \\ 1/100 & -1/100 \end{pmatrix}$$

with initial values $y_1(0) = 100$ and $y_2(0) = 0$.

b) Characteristic polynomial of $A$ is

$$\det(A - \lambda I) = (-1/100 - \lambda)(-1/100 - \lambda)$$

so there’s one eigenvalue: $\lambda = -1/100$. The eigenvector is $v_1 = (0, 1)^t$.

The generalized eigenvector $v_2$ is given by solving $(A + 1/100I)v_2 = v_1$.

Solving yields $v_2 = (100, 0)^t$. So the general solution is

$$e^{-t/100}(C_1v_1 + C_2(tv_1 + v_2))$$

Plugging in $t = 0$ gives $C_1v_1 + C_2v_2 = (100, 0)^t$. Solving gives $C_1 = 0$ and $C_2 = 1$. Thus the solution satisfying the initial conditions is

$$e^{-t/100}(tv_1 + v_2) = e^{-t/100}\begin{pmatrix} 100 \\ t \end{pmatrix}.$$